Controlled KK-theory I: a Milnor exact sequence

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Abstract

We introduce controlled KK-theory groups associated to a pair (A, B) of separable C^*-algebras. Roughly, these consist of elements of the usual K-theory group $K_0(B)$ that approximately commute with elements of A. Our main results show that these groups are related to Kasparov's KK-groups by a Milnor exact sequence, in such a way that Rørdam's KL-group is identified with an inverse limit of our controlled KK-groups.

In the case that the C^*-algebras involved satisfy the UCT, our Milnor exact sequence agrees with the Milnor sequence associated to a KK-filtration in the sense of Schochet, although our results are independent of the UCT. Applications to the UCT will be pursued in subsequent work.

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1 Introduction

Given two C^*-algebras A and B, Kasparov associated an abelian group $KK(A, B)$ of generalized morphisms between A and B. The Kasparov KK-groups were designed to have applications to index theory and the Novikov conjecture [14], but now play a fundamental role in many aspects of C^*-algebra theory (and elsewhere). This is particularly true in the Elliott program [9] to classify C^*-algebras by K-theoretic invariants.

Our immediate goal in this paper is to introduce *controlled KK-theory groups* and relate them to Kasparov's KK-theory groups. The idea – which we will pursue in subsequent work – is that the controlled groups allow more flexibility in computations. Our groups are analogues of the controlled K-theory groups introduced by the second author as part of his work on the Novikov conjecture [31], and later developed by him in collaboration with Oyono-Oyono [16]. Having said that, our approach in this paper is independent of, and in some sense dual to, these earlier developments: controlled K-theory abstracts the approach to the Novikov conjecture through operators of controlled propagation, while the controlled KK-theory we introduce here abstracts the dual approach to the Novikov conjecture through almost flat bundles (see for example [2] and [28, Chapter 11]).

Our larger goal is to establish a new sufficient condition for a nuclear C^*-algebra to satisfy the UCT of Rosenberg and Schochet [19], analogously to recent results on the Künneth formula using controlled K-theory ideas [17, 27]. The applications will come in the companion paper [29]. Our goal in this paper is to establish the basic theory, which we hope will be useful in other settings.

Controlled KK-theory and the Milnor sequence

We now discuss a version of our controlled KK-theory groups in more detail.

Let B be a separable C^*-algebra, let $B \otimes \mathcal{K}$ be its stabilization, and let $M(B \otimes \mathcal{K})$ be its stable multiplier algebra. Define $\mathcal{P}(B)$ to consist of all projections in $p \in M_2(M(B \otimes \mathcal{K}))$ such that $p - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is in the ideal $M_2(B \otimes \mathcal{K})$. Then the formula

$$\pi_0(\mathcal{P}(B)) \to K_0(B), \quad [p] \mapsto [p] - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{1}$$

gives a bijection from the set of path components of $\mathcal{P}(B)$ to the usual K_0-group of B.

Now, assume for simplicity that A is a separable, unital, and nuclear[1] C^*-algebra. Let $\pi : A \to \mathcal{B}(\ell^2)$ be an infinite amplification of a faithful unital representation[2], and use the composition

$$A \to \mathcal{B}(\ell^2) = M(\mathcal{K}) \subseteq M(B \otimes \mathcal{K})$$

of π and the canonical inclusion of $M(\mathcal{K})$ into $M(B \otimes \mathcal{K})$ to consider A as a C^*-subalgebra of $M(B \otimes \mathcal{K})$. Having A act as diagonal matrices, we may also identify A with a C^*-subalgebra of $M_2(M(B \otimes \mathcal{K}))$. For a finite subset X of A and $\epsilon > 0$, define

$$\mathcal{P}_\epsilon(X, B) := \{p \in \mathcal{P}(B) \mid \|[p, a]\| < \epsilon \text{ for all } a \in X\}.$$

[1] The assumptions of unitality and nuclearity are not necessary, but simplify the definitions - see the body of the paper for the general versions.

[2] i.e. take a faithful unital representation on a separable Hilbert space, and add it to itself countable many times. It turns out the choices involved here do not matter in any serious way.

Define the *controlled KK-theory group*[3] *associated to X and ϵ* to be

$$KK_\epsilon(X,B) := \pi_0(\mathcal{P}_\epsilon(X,B)).$$

Thanks to the isomorphism of line (1), we think of $KK_\epsilon(X,B)$ as "the part of $K_0(B)$ that commutes with X up to ϵ". This idea – of considering elements of K-theory that asymptotically commute with some representation – is partly inspired by the E-theory of Connes and Higson [3].

Now, let (X_n) be a nested sequence of finite subsets of A with dense union, and let (ϵ_n) be a decreasing sequence of positive numbers than tend to zero. As it is easier to commute with X_n up to ϵ_n that it is to commute with X_{n+1} up to ϵ_{n+1}, we get a sequence of "forgetful" homomorphisms

$$\cdots \to KK_{\epsilon_n}(X_n,B) \to KK_{\epsilon_{n-1}}(X_{n-1},B) \to \cdots \to KK_{\epsilon_1}(X_1,B).$$

Thus we may build the inverse limit $\varprojlim KK_{\epsilon_n}(X_n,B)$ of abelian group theory associated to this sequence. Replacing B with its suspension SB, we may also build the $\lim^1$-group[4] $\varprojlim^1 KK_{\epsilon_n}(X_n,SB)$ associated to the corresponding sequence. We are now ready to state a special case of our main theorem.

Theorem 1.1. *For any separable C^*-algebras A and B with A unital and nuclear*[5]*, there is a short exact sequence*

$$0 \longrightarrow \varprojlim{}^1 KK_{\epsilon_n}(X_n,SB) \longrightarrow KK(A,B) \longrightarrow \varprojlim KK_{\epsilon_n}(X_n,B) \longrightarrow 0\,.$$

We will explain the idea of the proof below, but first give a more precise version involving Rørdam's KL-groups [18, Section 5], and some comparisons of the results with the previous literature.

[3] It is a group in a natural way, in a way that is compatible with the group structure on $K_0(B)$.

[4] The $\lim^1$-group is constructed using the first derived functor of the inverse limit functor: see for example [26, Section 3.5]. See for example [26, Definition 3.5.1] for concrete definitions of the inverse limit and $\lim^1$ groups.

[5] There is also a very similar version for general separable A: see the main body of the paper

The topology on KK and Schochet's Milnor sequence

Recall that $KK(A,B)$ is equipped with a canonical topology, which makes it a (possibly non-Hausdorff) topological group. This topology can be described in several different ways that turn out to be equivalent, as established by Dadarlat in [5] (see also [21]). We define[6] $KL(A,B)$ to be to be the associated 'Hausdorffification', i.e. the quotient $KK(A,B)/\overline{\{0\}}$ of $KK(A,B)$ by the closure of the zero element.

The following theorem relating our controlled KK-theory groups to the topology on KK is a more precise version of Theorem 1.1, and is what we actually establish in the main body of the paper.

Theorem 1.2. *For any separable C^*-algebras A and B with A unital and nuclear*[7]*, there are canonical isomorphisms*

$$\varprojlim{}^1 KK_{\epsilon_n}(X_n, SB) \cong \overline{\{0\}} \quad \textit{and} \quad \varprojlim KK_{\epsilon_n}(X_n, B) \cong KL(A,B).$$

The short exact sequence in Theorem 1.1 is an analogue of Schochet's *Milnor exact sequence* [20] associated to a *KK-filtration*. A KK-filtration consists of a KK-equivalence of A with the direct limit of an increasing sequence (A_n) of C^*-algebras where each A_n has unitization the continuous functions on some finite CW complex. Schochet [20, Theorem 1.5] shows that such a filtration exists if and only if A satisfies the UCT. Schochet [20, Theorem 1.5] then proves that there is an exact sequence

$$0 \longrightarrow \varprojlim{}^1 KK(A_n, SB) \longrightarrow KK(A,B) \longrightarrow \varprojlim KK(A_n, B) \longrightarrow 0\,.$$

It follows from Theorem 1.2 and [22, Proposition 4.1] that our Milnor sequence from Theorem 1.1 agrees with Schochet's when A satisfies the UCT. Our Milnor sequence can thus be thought of as a generalization of Schochet's sequence that works in the absence of the UCT.

[6] The original definition of $KL(A,B)$ is due to Rørdam [18, Section 5], and only makes sense if the pair (A,B) satisfies the UCT. The definition we are using was suggested by Dadarlat [5, Section 5], and is equivalent to Rørdam's when A satisfies the UCT by [5, Theorem 4.1].

[7] Again, these extra assumptions on A are not necessary, with minor changes to the definitions.

Discussion of proofs

Continuing to assume for simplicity that A is unital and nuclear, let us identify A with a C^*-subalgebra of $M(B\otimes\mathcal{K})$ as in the statement of Theorem 1.1. Then we define $\mathcal{P}(A,B)$ to be the collection of all continuous, bounded, projection-valued functions $p:[1,\infty)\to M_2(M(B\otimes\mathcal{K}))$ such that $[p_t,a]\to 0$ as $t\to\infty$ for all $a\in A$, and so that $p_t-\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ is in $M_2(B\otimes\mathcal{K})$ for all t. Define $KK_{\mathcal{P}}(A,B)$ to be the quotient of $\mathcal{P}(A,B)$, modulo the equivalence relation one gets by saying p_0 and p_1 are homotopic if they are restrictions to the endpoints of an element of $\mathcal{P}(A,C[0,1]\otimes B)$.

One can then show[8] that $KK(A,B)$ is naturally isomorphic to $KK_{\mathcal{P}}(A,B)$. The first important ingredient in this is the description of $KK(A,B)$ as the K-theory of an appropriate *localization algebra*, which was done by Dadarlat, Wu and the first author in [8, Theorem 4.4] (inspired by ideas of the second author in the case of commutative C^*-algebras [30]). The other important (albeit implicit) ingredient we use for the isomorphism $KK(A,B)\cong KK_{\mathcal{P}}(A,B)$ is the fundamental theorem of Kasparov ([13, Section 6, Theorem 1], and see also [23, Theorem 19] and [1, Section 18.5]) that the equivalence relations on Kasparov cycles induced by operator homotopy and homotopy give rise to the same KK-groups.

Having described $KK(A,B)$ using continuous paths of projections, we can now also describe the topology on this group in this language: roughly, a sequence (p^n) converges to p in $\mathcal{P}(A,B)$ if for all $\epsilon>0$ and finite $X\subseteq A$ there is t_0 such that for all $t\geqslant t_0$, p_t^n can be connected to p_t via a homotopy passing through $\mathcal{P}_\epsilon(X,B)$. This topology on $\mathcal{P}(A,B)$ induces a topology on $KK_{\mathcal{P}}(A,B)$, and we show that this topology agrees with the usual one on $KK(A,B)$ using an abstract characterization of the latter due to Dadarlat [5, Section 3].

Having got this far, it is not too difficult to see that there is a well-defined

[8]We do not actually show this, only the more general version where A is non-unital and not-necessarily nuclear; nonetheless, this result directly follows from the same methods.

map

$$KK_{\mathcal{P}}(A,B) \to \lim_{\leftarrow} KK_{\epsilon}(X,B) \tag{2}$$

defined by evaluating a path $(p_t)_{t\in[1,\infty)}$ in $\mathcal{P}(A,B)$ at larger and larger values of t, and that there is a well-defined map

$$\lim_{\leftarrow}{}^1 KK_{\epsilon}(X,SB) \to KK_{\mathcal{P}}(A,B) \tag{3}$$

defined by treating an element of $KK_{\epsilon}(X,SB)$ as a projection-valued function from $[0,1]$ to $KK_{\epsilon}(X,B)$ that agrees with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ at its endpoints, and stringing a countable sequence of these together to get an element $(p_t)_{t\in[1,\infty)}$ in $\mathcal{P}(A,B)$. Moreover, it is essentially true by definition that the map in line (2) contains the closure of $\{0\}$ in its kernel, and the map in line (3) takes image if the closure of $\{0\}$. To establish Theorem 1.2, we show that these maps are both isomorphisms.

Notation and conventions

We write ℓ^2 for $\ell^2(\mathbb{N})$.

Throughout, the symbols A and B are reserved for separable C^*-algebras (the letters C, D and others may sometimes refer to non-separable C^*-algebras). The unit ball of a C^*-algebra C is denoted by C_1, its unitization is C^+, and its multiplier algebra is $M(C)$.

Our conventions on Hilbert modules follow those of Lance [15]. We will always assume that Hilbert modules are over separable C^*-algebras, and are countably generated as discussed on [15, page 60]. If it is not explicitly specified otherwise, all Hilbert modules will be over the C^*-algebra called B. For Hilbert B-modules E and F, we write $\mathcal{L}(E,F)$ (respectively $\mathcal{K}(E,F)$) for the spaces of adjointable (respectively compact) operators from E to F in the usual sense of Hilbert module theory [15, pages 8-10]. We use the standard shorthands $\mathcal{L}(E) := \mathcal{L}(E,E)$ and $\mathcal{K}(E) := \mathcal{K}(E,E)$. In this paper, a *representation of* A will always refer to a representation of A on a Hilbert *module*, i.e. a $*$-homomorphism $\pi : A \to \mathcal{L}(E)$ for some Hilbert module E (almost always over B, as above). We write E^{∞} for the (completed) infinite direct sum Hilbert module $\bigoplus_{n=1}^{\infty} E = \ell^2 \times E$, and if $\pi : A \to \mathcal{L}(E)$ is a representation, we write $\pi^\infty : A \to \mathcal{L}(E^\infty)$ for the amplified representation, so in tensor product language $\pi^\infty = 1_{\ell^2} \otimes \pi : A \to \mathcal{L}(\ell^2 \otimes E)$. We say a representation (π, E) has *infinite multiplicity* if it isomorphic to $(\sigma^\infty, F^\infty)$ for some other representation (σ, F).

The symbol '$\otimes$' always denotes a completed tensor product: either the (external or internal) tensor product of Hilbert modules [15, Chapter 4], or the minimal tensor product of C^*-algebras.

If E is a Banach space and X a locally compact Hausdorff space, we let $C_b(X, E)$ (respectively, $C_{ub}(X, E)$, $C_0(X, E)$) denote the Banach space of continuous and bounded (respectively uniformly continuous and bounded, continuous and vanishing at infinity) functions from X to E. We write elements of these spaces as e or $(e_x)_{x\in X}$, with $e_x \in E$ denoting the value of e at a point $x \in X$. We will sometimes say that e is a '...' if it is a pointwise a '...': for example, '$u \in C_b([1, \infty), \mathcal{L}(F_1, F_2))$ is unitary' means 'u_t is unitary in $\mathcal{L}(F_1, F_2)$ for all $t \in [1, \infty)$'; if E is a C^*-algebra and $e \in C_b(X, E)$, this is consistent with the standard use of 'unitary' and so on. With u as above, if b is an element of $\mathcal{L}(F_1)$ we write ub for the function $t \mapsto u_t b$ in $C_{ub}([1, \infty), \mathcal{L}(F_1, F_2))$ and similarly for cu with $c \in \mathcal{L}(F_2)$ and so on.

For K-theory, $K_*(A) := K_0(A) \oplus K_1(A)$ denotes the graded K-theory group of a C^*-algebra, and $KK_*(A, B) := KK_0(A, B) \oplus KK_1(A, B)$ the graded KK-theory group. We will typically just write $KK(A, B)$ instead of $KK_0(A, B)$.

Outline of the paper

Sections 2 and 3 are background. In Section 2 we recall some basic facts about so-called absorbing representation, and prove some basic results. Most of the material in Section 2 is essentially from papers of Thomsen [25], Dadarlat-Eilers [6, 7], and Dadarlat [5]: we do not claim any real originality. In Section 3 we recall the localization algebra of Dadarlat, Wu and the first author [8] (inspired by much earlier ideas of the second author [30]), and prove some technical results about this.

Sections 4 and 5 introduce a group $KK_{\mathcal{P}}(A, B)$ that consists of (homotopy classes) of projections that asymptotically commute with A and relate it to KK-theory: the culminating results show that $KK(A, B)$ and $KK_{\mathcal{P}}(A, B)$ are isomorphic as topological groups. In Section 4 we introduce $KK_{\mathcal{P}}(A, B)$, show that it is a commutative monoid, and then that it is isomorphic to $KK(A, B)$ (whence a group). In Section 5 we introduce a topology on $KK_{\mathcal{P}}(A, B)$. We then use a characterization of Dadarlat [5, Section 3] to identify this with the canonical topology on $KK(A, B)$ that was introduced and studied by Brown-Salinas, Schochet, Pimsner, and Dadarlat in various guises.

Sections 6 and 7 establish Theorem 1.2 (and therefore Theorem 1.1). Section 6 identifies the quotient $KK_{\mathcal{P}}(A, B)/\overline{\{0\}}$ with $\varprojlim KK_{\epsilon}(X, B)$ (and therefore identifies $KL(A, B)$ with this inverse limit). Section 7 identifies the closure of zero in $KK_{\mathcal{P}}(A, B)$ with the appropriate $\lim^1$ group, completing the proof of the main results.

Finally, Appendix A gives some alternative pictures of our controlled KK-groups that will be useful for our subsequent work.

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2 Strongly absorbing representations

Throughout this section, A and B refer to separable C^*-algebras.

In this section we establish conventions and terminology regarding representations on Hilbert modules. The ideas in this section are not original: the original sources are papers of Thomsen [25], Dadarlat [5], Dadarlat-Eilers [7], and Kasparov [12]. Nonetheless, we need variations of the material appearing in the literature, so record what we need here for the reader's convenience and provide proofs where a precise result has not appeared before.

The definition of absorbing representation below is essentially[9] due to Thomsen [25, Definition 2.6].

Definition 2.1. A representation $\pi : A \to \mathcal{L}(F)$ is *absorbing* (for the pair (A, B)) if for any Hilbert B-module E and ccp map $\sigma : A \to \mathcal{L}(E)$, there is a sequence (v_n) of isometries in $\mathcal{L}(E, F)$ such that:

(i) $\sigma(a) - v_n^*\pi(a)v_n \in \mathcal{K}(E)$ for all $a \in A$ and $n \in \mathbb{N}$;

(ii) $\|\sigma(a) - v_n^*\pi(a)v_n\| \to 0$ as $n \to \infty$ for all $a \in A$.

We want something slightly stronger.

Definition 2.2. A representation $\pi : A \to \mathcal{L}(F)$ is *strongly absorbing* (for the pair (A, B)) if (π, F) is the infinite amplification $(\sigma^\infty, E^\infty)$ of an absorbing representation (σ, E).

Remark 2.3. If (π, F) is an infinite multiplicity (for example, strongly absorbing) representation then there we can write it as an infinite direct sum of copies of itself. It follows that there is a sequence $(s_n)_{n=1}^\infty$ of isometries in $\mathcal{L}(F)$ with mutually orthogonal ranges, that commute with the image of

[9] Thomsen's definition is a little more restrictive: he insists that B be stable, and that the B-modules used all be copies $B \otimes \mathcal{K}(\ell^2)$. Thanks to a combination of Kasparov's stabilization theorem [12, Theorem 2] and Remark A.16 below, our extra generality makes no real difference.

the representation, and have sum $\sum_{n=1}^{\infty} s_n s_n^*$ that converges strictly[10] to the identity.

In [25, Theorem 2.7], Thomsen shows that an absorbing representation of A on $\ell^2 \otimes B$ always exists. The following is therefore immediate from the fact that $(\ell^2 \otimes B)^\infty \cong \ell^2 \otimes B$.

Proposition 2.4. *There is a strongly absorbing representation of A on $\ell^2 \otimes B$.* □

The point of using strongly absorbing representations rather than just absorbing[11] ones is to get the following lemma.

Lemma 2.5. *Let $\pi : A \to \mathcal{L}(F)$ be a strongly absorbing representation, and let $\sigma : A \to \mathcal{L}(E)$ be a ccp map. Then there is a sequence (v_n) of isometries in $\mathcal{L}(E, F)$ such that:*

(i) $\sigma(a) - v_n^* \pi(a) v_n \in \mathcal{K}(E, F)$ *for all* $a \in A$ *and* $n \in \mathbb{N}$*;*

(ii) $\|\sigma(a) - v_n^* \pi(a) v_n\| \to 0$ *as* $n \to \infty$ *for all* $a \in A$*;*

(iii) $v_n^* v_m = 0$ *for all* $n \neq m$*.*

Proof. Let $(\pi, F) = (\theta^\infty, G^\infty)$ for some absorbing representation (θ, G). Let (w_n) be a sequence of isometries in $\mathcal{L}(E, G)$ with the properties as in the definition of an absorbing representation for σ. For each n, let $s_n \in \mathcal{L}(G, F)$ be the inclusion of G in F as the n^{th} summand, and set $v_n := s_n w_n \in \mathcal{L}(E, F)$. It is straightforward to check that (v_n) has the right properties. □

We will need the following result, which is implicit[12] in [6].

[10] For the strict topology coming from the identification $\mathcal{L}(F) = M(\mathcal{K}(F))$. As the partial sums are uniformly bounded, we may equivalently use the topology of pointwise convergence as operators on F.

[11] We do not know that the lemma fails for absorbing representations, but cannot prove it either.

[12] It is also explicit in [8, Theorem 2.6], but with π only assumed absorbing, not strongly absorbing. There seems to be a gap in the proof of that result. As a result, it seems to be necessary to assume all absorbing modules used in [8] are actually strongly absorbing. None of the results of [8] are further affected if one does this.

Proposition 2.6. *Let $\pi : A \to \mathcal{L}(E)$ be a strongly absorbing representation. Then for any ccp map $\sigma : A \to \mathcal{L}(F)$ there is an isometry $v \in C_{ub}([1,\infty), \mathcal{L}(F,E))$ such that $v^*\pi(a)v - \sigma(a) \in C_0([1,\infty), \mathcal{K}(F))$.*

Moreover, if $\sigma : A \to \mathcal{L}(F)$ is also a strongly absorbing representation, then there is a unitary $u \in C_{ub}([1,\infty), \mathcal{L}(F,E))$ such that $u^*\pi(a)u - \sigma(a) \in C_0([1,\infty), \mathcal{K}(F))$.

We will need two lemmas. The first is a well-known algebraic trick.

Lemma 2.7. *Let $\pi : A \to \mathcal{L}(E)$ and $\sigma : A \to \mathcal{L}(F)$ be representations, and $v \in \mathcal{L}(E,F)$ be an isometry. If $v \in C_{ub}([1,\infty), \mathcal{L}(F,E))$ is such that $v^*\pi(a)v - \sigma(a) \in C_0([1,\infty), \mathcal{K}(F))$ for all $a \in A$, then $\pi(a)v - v\sigma(a)$ is an element of $C_0([1,\infty), \mathcal{K}(F,E))$ for all $a \in A$.*

Proof. This follows from the fact that

$$(\pi(a)v - v\sigma(a))^*(\pi(a)v - v\sigma(a))$$

equals

$$v^*\pi(a^*a)v - \sigma(a^*a) - (v^*\pi(a^*)v - \sigma(a^*))\sigma(a) - \sigma(a^*)(v^*\pi(a)v - \sigma(a))$$

for all $a \in A$. □

The second lemma we need is [7, Lemma 2.16]; we recall the statement for the reader's convenience but refer to the reference for a proof.

Lemma 2.8. *Let $\pi : A \to \mathcal{L}(E)$ and $\sigma : A \to \mathcal{L}(F)$ be representations. Let $\sigma^\infty : A \to \mathcal{L}(F^\infty)$ be the infinite amplification of E, and let $w \in \mathcal{L}(F^\infty, F \oplus F^\infty)$ be defined by $(\xi_1, \xi_2, \xi_3 ...) \mapsto \xi_1 \oplus (\xi_2, \xi_3, ...)$. Then for any isometry $v \in \mathcal{L}(F^\infty, E)$, the operator*

$$u := (1_F \oplus v)wv^* + 1_E - vv^* \in \mathcal{L}(E, F \oplus E)$$

is unitary and satisfies

$$\|\sigma(a) \oplus \pi(a) - u\pi(a)u^*\| \leqslant 6\|v\sigma^\infty(a) - \pi(a)v\| + 4\|v\sigma^\infty(a^*) - \pi(a^*)v\|.$$

Moreover, if $v\sigma^\infty(a) - \pi(a)v \in \mathcal{K}(F^\infty, E)$ for all $a \in A$, then $\sigma(a) \oplus \pi(a) - u\pi(a)u^* \in \mathcal{K}(F \oplus E)$ for all $a \in A$. □

Proof of Proposition 2.6. Say first $\sigma : A \to \mathcal{L}(F)$ is ccp. Let (v_n) be a sequence of isometries in $\mathcal{L}(F, E)$ as Lemma 2.5. For each $n \geqslant 1$ and each $t \in [n, n+1]$, define

$$v_t := (n+1-t)^{1/2} v_n + (t-n)^{1/2} v_{n+1}$$

Then the resulting family $v := (v_t)_{t \in [1,\infty)}$ is an isometry in $C_{ub}([1,\infty), \mathcal{L}(F, E))$ such that $v^*\pi(a)v - \sigma(a) \in C_0([1,\infty), \mathcal{K}(F))$ for all $a \in A$; we leave the direct checks involved to the reader.

Assume now that $\sigma : A \to \mathcal{L}(F)$ is also a strongly absorbing representation. Using the first part of the proof applied to the infinite amplification $\sigma^\infty : A \to \mathcal{L}(F^\infty)$, we get $v \in C_{ub}([1,\infty), \mathcal{L}(F^\infty, E))$ such that $v^*\pi(a)v - \sigma^\infty(a) \in C_0([1,\infty), \mathcal{K}(F^\infty))$ for all $a \in A$. Lemma 2.7 implies that $\pi(a)v - v\sigma^\infty(a)$ is an element of $C_0([1,\infty), \mathcal{K}(F^\infty, E))$ for all $a \in A$. Building a unitary out of each v_t using the formula in Lemma 2.8 gives now a unitary $u_E \in C_{ub}([1,\infty), \mathcal{L}(E, F \oplus E))$ such that $\sigma(a) \oplus \pi(a) - u_E\pi(a)u_E^* \in C_0([1,\infty), \mathcal{K}(F \oplus E))$ for all $a \in A$. The situation is symmetric, so there is also a unitary $u_F \in C_{ub}([1,\infty), \mathcal{L}(F, F \oplus E))$ such that $\sigma(a) \oplus \pi(a) - u_F\pi(a)u_F^* \in C_0([1,\infty), \mathcal{K}(F \oplus E))$ for all $a \in A$. Defining $u = u_E^* u_F$, we are done. □

We need one more technical result about strongly absorbing representations. The statement and proof are essentially[13] the same as a result of Dadarlat [5, Proposition 3.2]. We give give a proof for the reader's convenience.

Proposition 2.9. *Let $\pi : A \to \mathcal{L}(B \otimes \ell^2)$ be a strongly absorbing representation of A on the standard Hilbert B-module. Let C be a separable nuclear C^*-algebra, and let $C \otimes B \otimes \ell^2$ denote the $C \otimes B$-Hilbert module given by the exterior tensor product. Then the amplification $1_C \otimes \pi : A \to \mathcal{L}(C \otimes B \otimes \ell^2)$ is strongly absorbing for the pair $(A, C \otimes B)$.*

Proof. As $1_C \otimes \pi$ is isomorphic to the infinite amplification of itself, it suffices to prove that $1_C \otimes \pi$ is absorbing. Let $(1_C \otimes \pi)^+ : A^+ \to \mathcal{L}(C \otimes B \otimes \ell^2)$ be the

[13] The main difference is that we drop a unitality assumption on C. This will be useful below.

canonical unital extension of $1_C \otimes \pi$ to the unitization A^+ of A (even if A is already unital). Using Kasparov's stabilization theorem [12, Theorem 2], the equivalence of (1) and (2) from [25, Theorem 2.5], [25, Theorem 2.1], and the canonical identifications $C \otimes B \otimes \mathcal{K}(\ell^2) = \mathcal{K}(C \otimes B \otimes \ell^2)$ and $\mathcal{L}(C \otimes B \otimes \ell^2) = M(\mathcal{K}(C \otimes B \otimes \ell^2))$, it suffices to show that if $\sigma : A^+ \to C \otimes B \otimes \mathcal{K}(\ell^2)$ is any ccp map then there is a sequence (w_n) in $\mathcal{L}(C \otimes B \otimes \mathcal{K}(\ell^2))$ such that

$$\lim_{n\to\infty} \|\sigma(a) - w_n^*(1 \otimes \pi)^+(a) w_n\| = 0 \quad \text{for all} \quad a \in A^+$$

and such that

$$\lim_{n\to\infty} \|w_n^* b\| = 0 \quad \text{for all} \quad b \in C \otimes B \otimes \mathcal{K}(\ell^2).$$

Let $\delta : C^+ \to \mathcal{B}(\ell^2)$ be a unital representation of the unitization of C such that $\delta^{-1}(\mathcal{K}(\ell^2)) = \{0\}$. Let $\iota : C^+ \to \mathcal{L}(C)$ be the canonical multiplication representation. Kasparov's version of Voiculescu's theorem [12, Theorem 5] combined with nuclearity of C^+ imply that there is a sequence $(v_{n,(0)})_{n=1}^{\infty}$ of isometries in $\mathcal{L}(C, C^+ \otimes \ell^2)$ such that

$$\|\iota(c) - v_{n,(0)}^* \big(1_{C^+} \otimes \delta(c)\big) v_{n,(0)}\| \to 0 \quad \text{as} \quad n \to \infty$$

for all $c \in C^+$. Perturbing $v_{n,(0)}$ slightly, we may assume that actually $v_{n,(0)}$ has image in $C^+ \otimes \ell^2\{1, ..., k(n)\}$ for some $k(n)$.

Let (e_m) be an approximate unit for C. We may consider multiplication by e_m as defining an operator in $\mathcal{L}_C(C^+ \otimes H, C \otimes H))$ for any Hilbert space H, and therefore the product operators $e_m v_{n,(0)}$ make sense in $\mathcal{L}(C, C \otimes \ell^2\{1, ..., k(n)\})$. For a suitable choice of $m(n)$ we have that if $v_{n,(1)} := e_{m(n)} v_{n,(0)}$ then

$$\|\iota(c) - v_{n,(1)}^* \big(1_C \otimes \delta(c)\big) v_{n,(1)}\| \to 0 \quad \text{as} \quad n \to \infty$$

for all $c \in C$. Let $\delta_n : C \to \mathcal{K}(\ell^2\{1, ..., k(n)\})$ be the compression of δ to the first n basis vectors. Note that by choice of $k(n)$ we have $v_{n,(1)}^* \big(1_C \otimes \delta(c)\big) v_{n,(1)} = v_{n,(1)}^* \big(1_C \otimes \delta_n(c)\big) v_{n,(1)}$ for all n, and thus that

$$\|\iota(c) - v_{n,(1)}^* \big(1_C \otimes \delta_n(c)\big) v_{n,(1)}\| \to 0 \quad \text{as} \quad n \to \infty$$

for all $c \in C$.

Define

$$\Delta_n := \delta_n \otimes 1_{B\otimes\mathcal{K}(\ell^2)} : C \otimes B \otimes \mathcal{K}(\ell^2) \to \mathcal{K}(\ell^2\{1, ..., k(n)\}) \otimes B \otimes \mathcal{K}(\ell^2).$$

Define $v_{n,(2)} := v_{n,(1)} \otimes 1_{B\otimes\mathcal{K}(\ell^2)}$, so

$$v_{n,(2)} \in \mathcal{L}_{C\otimes B\otimes\mathcal{K}(\ell^2)}\big(C \otimes B \otimes \mathcal{K}(\ell^2)\,,\, C \otimes \ell^2\{1, ..., k(n)\} \otimes B \otimes \mathcal{K}(\ell^2)\big).$$

Note then that

$$\|c - v_{n,(2)}^*\big(1_C \otimes \Delta_n(c)\big)v_{n,(2)}\| \to 0 \quad \text{as} \quad n \to \infty$$

for all $c \in C \otimes B \otimes \mathcal{K}(\ell^2)$ and so in particular

$$\|\sigma(a) - v_{n,(2)}^*\big(1_C \otimes \Delta_n(\sigma(a))\big)v_{n,(2)}\| \to 0 \quad \text{as} \quad n \to \infty$$

for all $a \in A^+$.

To complete the proof, use an identification $\mathcal{K}(\ell^2\{1, ..., k(n)\}) \otimes \mathcal{K}(\ell^2) \cong \mathcal{K}(\ell^2)$ to give an isomorphism $\phi : \mathcal{K}(\ell^2\{1, ..., k(n)\}) \otimes B \otimes \mathcal{K}(\ell^2) \to B \otimes \mathcal{K}(\ell^2)$. Note that as π is absorbing there is a sequence $(v_{n,(3)})_{n=1}^\infty$ in $\mathcal{L}(B \otimes \mathcal{K}(\ell^2))$ such that

$$\|\phi(\Delta_n(\sigma(a))) - v_{n,(3)}^*\pi(a)v_{n,(3)}\| \to 0 \quad \text{as} \quad n \to \infty$$

for all $a \in A^+$ (compare the equivalence of (1) and (2) from [25, Theorem 2.5] again). As π is strongly absorbing, we may moreover assume that the $v_{n,(3)}$ satisfy $v_{n,(3)}^*b \to$ for all $b \in B \otimes \mathcal{K}(\ell^2)$ by ensuring that for all m, $v_{n,(3)}v_{n,(3)}^*$ is orthogonal to any element of $B \otimes \mathcal{K}(\ell^2\{1, ..., m\})$ for all large n. It is then not too difficult to check that we can choose $l(n)$ such that if we set

$$w_n := (1_C \otimes v_{l(n),(3)})v_{n,(2)} \in \mathcal{L}(C \otimes B \otimes \mathcal{K}(\ell^2)),$$

then (w_n) has the right properties. $\square$

3 Localization algebras

As usual, A and B refer to separable C^*-algebras throughout this section.

In this section, we define localization algebras following [8], and show that uniform continuity can be replaced with continuity in the definition without changing the K-theory. This result was first observed by Jianchao Wu (with a different proof), and we thank him for permission to include it here.

The following definition comes from [8, Section 3]. We use slightly different notation to that reference to differentiate between the continuous and uniformly continuous versions.

Definition 3.1. Let $\pi : A \to \mathcal{L}(E)$ be a representation. Define $C_{L,u}(\pi)$ to be the C^*-algebra of all bounded, uniformly continuous functions $b : [1, \infty) \to \mathcal{L}(E)$ such that $[b_t, a] \to 0$ for all $a \in A$, and such that ab_t is in $\mathcal{K}(E)$ for all $a \in A$ and all $t \in [1, \infty)$. We call $C_{L,u}(\pi)$ the *localization algebra* of π.

The following is[14] [8, Theorem 4.4].

Theorem 3.2. *Let $\pi : A \to \mathcal{L}(E)$ be a strongly absorbing representation. Then there is a canonical isomorphism $KK_*(A, B) \to K_*(C_{L,u}(\pi))$.* □

Definition 3.3. Let $\pi : A \to \mathcal{L}(E)$ be a representation of A on a Hilbert B-module. Define $C_{L,c}(\pi)$ to be the C^*-algebra of all bounded, continuous functions $b : [1, \infty) \to \mathcal{L}(E)$ such that $[b_t, a] \to 0$ for all $a \in A$, and such that ab_t is in $\mathcal{K}(E)$ for all $a \in A$ and all $t \in [1, \infty)$.

Clearly there is a canonical inclusion $C_{L,u}(\pi) \to C_{L,c}(\pi)$. Our main goal in this section is to prove the following result.

Theorem 3.4. *Let $\pi : A \to \mathcal{L}(F)$ be an infinite multiplicity (in particular, strongly absorbing) representation of A on a Hilbert B-module. Then the canonical inclusion $C_{L,u}(\pi) \to C_{L,c}(\pi)$ induces an isomorphism on K-theory.*

We will need two preliminary lemmas. The first of these follows from standard techniques: compare for example [11, Proposition 4.1.7].

[14] As explained in footnote 12, the cited result should be stated with the assumption that the representation is strongly absorbing, not just absorbing.

Lemma 3.5. *There is a function $\omega : [0,\infty) \to [0,\infty)$ such that $\omega(0) = 0$, such that $\lim_{t\to 0} \omega(t) = 0$, and with the following property. Say D is a C^*-algebra, and say $p^0, p^1 \in C([0,1], D)$ are projections with the property that $\|p^0_t - p^1_t\| < 1/2$ for all $t \in [0,1]$. Then there is a homotopy $(p^s)_{s\in[0,1]}$ connecting them, and with the property that $\|p^s - p^{s'}\| \leqslant \omega(|s-s'|)$ for all distinct $s, s' \in [0,1]$*

Proof. We will work in $C([0,1], D^+)$, where D^+ is the unitization of D. Fix $t \in [0,1]$, and define

$$x_t := p^1_t p^0_t + (1-p^1_t)(1-p^0_t).$$

Then one checks as in the proof of [11, Proposition 4.1.7] that $\|1 - x_t\| < 1/2$. For $s \in [0,1]$, define $x^s_t := s1 + (1-s)x_t$, which also satisfies $\|1 - x^s_t\| < 1/2$. Hence each x^s_t is invertible, and the norm of its inverse is at most 2 by the usual Neumann series representation. Define moreover $u^s_t := x^s_t((x^s_t)^* x^s_t)^{-1/2}$, which is unitary. One computes as in the proof of [11, Proposition 4.1.7] that

$$u^1_t p^0_t (u^1_t)^* = p^1_t.$$

It is then not difficult to check that defining $p^s_t := u^s_t p^0_t (u^s_t)^*$ gives a path $(p^s)_{s\in[0,1]}$ with the right property. □

For the statement of the next lemma, recall that if C and D are C^*-algebras equipped with surjections $\pi_C : C \to Q$ and $\pi_D : D \to Q$ to a third C^*-algebra Q, then the *pushout* is the C^*-algebra $P := \{(c,d) \in C \oplus D \mid \pi_C(c) = \pi_D(d)\}$. Such a pushout gives rise to a canonical *pushout square*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & C \arrow[d, "\pi_C"] \\ D \arrow[r, "\pi_D"'] & Q \end{tikzcd} \tag{4}$$

where the arrows out of P are the natural (surjective) coordinate projections.

See for example [28, Proposition 2.7.15] for a proof of the next result.

Lemma 3.6. *Given a pushout diagram as in line (4) above, there is a six-term exact sequence*

$$\begin{array}{ccccc} K_0(P) & \longrightarrow & K_0(C)\oplus K_0(D) & \longrightarrow & K_0(Q) \\ \uparrow & & & & \downarrow \\ K_1(Q) & \longleftarrow & K_1(C)\oplus K_1(D) & \longleftarrow & K_1(P) \end{array}$$

of K-theory groups. The diagram is natural for maps between pushout squares in the obvious sense. □

Before the proof of Theorem 3.4, we record two more well-known K-theoretic lemmas. See for example [28, Proposition 2.7.5 and Lemma 2.7.6] for proofs[15].

Lemma 3.7. *If $\alpha, \beta : C \to D$ are $*$-homomorphisms with orthogonal images, then $\alpha + \beta : C \to D$ is also a $*$-homomorphism, and $(\alpha+\beta)_* = \alpha_* + \beta_*$.* □

Lemma 3.8. *Let $\alpha, \beta : A \to B$ be $*$-homomorphisms, and assume that there is a partial isometry v in the multiplier algebra of B such that $\alpha(a)v^*v = \alpha(a)$ for all $a \in A$, and so that $v\alpha(a)v^* = \beta(a)$ for all $a \in A$. Then α and β induce the same maps on K-theory.* □

Proof of Theorem 3.4. Let $E := \bigsqcup_{n\geqslant 1}[2n, 2n+1]$ and $O := \bigsqcup_{n\geqslant 1}[2n-1, 2n]$, equipped with the restriction of the metric from $[1, \infty)$. Let $C_{L,u}(\pi; E)$ denote the collection of all bounded, uniformly continuous functions $b : E \to \mathcal{L}(F)$ such that $ab_t \in \mathcal{K}(F)$ for all $a \in A$, and such that $[a, b_t] \to 0$ as $t \to \infty$. Define $C_{L,u}(\pi; O)$ and $C_{L,u}(\pi; E \cap O)$ similarly, and define $C_{L,c}(\pi; E)$, $C_{L,c}(\pi; O)$, and $C_{L,c}(\pi; O \cap E)$ analogously, but with 'uniformly continuous' replaced by 'continuous'. Then we have a commutative diagram of pushout

[15] The statement of [28, Proposition 2.7.5] has a typo: vv^* should be v^*v where it appears there.

squares

$$\begin{tikzcd}
C_{L,u}(\pi) \arrow[rr] \arrow[dr] \arrow[dd] & & C_{L,u}(\pi;E) \arrow[dr] \arrow[dd] & \\
& C_{L,c}(\pi) \arrow[rr] \arrow[dd] & & C_{L,c}(\pi;E) \arrow[dd] \\
C_{L,u}(\pi;O) \arrow[rr] \arrow[dr] & & C_{L,u}(\pi;E\cap O) \arrow[dr] & \\
& C_{L,c}(\pi;O) \arrow[rr] & & C_{L,c}(\pi;E\cap O)
\end{tikzcd}$$

where the diagonal arrows are the canonical inclusions, and all the other arrows are the obvious restriction maps. Using Lemma 3.6 and the five lemma, it thus suffices to prove that the maps $C_{L,u}(\pi;G) \to C_{L,c}(\pi;G)$ induce isomorphisms on K-theory for $G \in \{E, O, E \cap O\}$. For $E \cap O$, which just equals $\mathbb{N} \cap [1, \infty)$, this is clear: the map is the identity on the level of C^*-algebras as there is no difference between continuity and uniform continuity in this case. The cases of E and O are essentially the same, so we just focus on E.

Let now $E_{\mathbb{N}} := E \cap 2\mathbb{N} = \{2, 4, 6, ...\}$ be the set of positive even numbers. Then we have a surjective $*$-homomorphism $C_{L,u}(\pi;E) \to C_{L,u}(\pi;E_{\mathbb{N}})$ defined by restriction, and similarly for $C_{L,c}$; write $C^0_{L,u}(\pi;E)$ and $C^0_{L,c}(\pi;E)$ for the respective kernels. Then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C^0_{L,u}(\pi;E) \arrow[r] \arrow[d] & C_{L,u}(\pi;E) \arrow[r] \arrow[d] & C_{L,u}(\pi;E_{\mathbb{N}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^0_{L,c}(\pi;E) \arrow[r] & C_{L,c}(\pi;E) \arrow[r] & C_{L,c}(\pi;E_{\mathbb{N}}) \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences where the vertical maps are the canonical inclusions. The right hand vertical map is the identity as there is no difference between continuity and uniform continuity for maps out of $E_{\mathbb{N}}$. Hence by the five lemma and the usual long exact sequence in K-theory, it suffices to show that the left hand vertical map induces an isomorphism on K-theory. For

$r \in [0,1]$ let us define a $*$-homomorphism $h_r : C^0_{L,u}(\pi; E) \to C^0_{L,u}(\pi; E)$ by the following prescription for $b \in C^0_{L,u}(\pi; E)$. For $t \in [2n, 2n+1]$, we set

$$(h_r b)_t := b_{2n+r(t-2n)}$$

(in other words, h_r contracts $[2n, 2n+1]$ to $\{2n\}$). Using uniform continuity, $(h_r)_{r\in[0,1]}$ is a null-homotopy of $C^0_{L,u}(\pi; E)$, and therefore $K_*(C^0_{L,u}(\pi; E)) = 0$. It thus suffices to show that $K_*(C^0_{L,c}(\pi; E)) = 0$, which we spend the rest of the proof doing.

We will focus on the case of K_0 (which is in any case all we use in this paper); the case of K_1 is similar. Take then an arbitrary element $x \in K_0(C^0_{L,c}(\pi; E))$, which we may represent by a formal difference $x = [p] - [1_k]$ where p is a projection in the $m \times m$ matrices $M_m(C^0_{L,c}(\pi; E)^+)$ over the unitization $C^0_{L,c}(\pi; E)^+$ of $C_{L,c}(\pi; E)$ for some m, and $1_k \in M_m(\mathbb{C}) \subseteq M_m(C^0_{L,c}(\pi; E)^+)$ is the scalar matrix with 1s in the first k diagonal entries and 0s elsewhere for some $k \leqslant m$. Without loss of generality may think of p as a continuous projection-valued function

$$p : E \to M_m(\mathcal{L}(F))$$

such that $a(p - 1_k) \in M_m(\mathcal{K}(F))$ for all $a \in A$ (here we use the amplification of the representation of A to a representation on $M_m(\mathcal{L}(F))$ to make sense of this), such that $[a, p_t] \to 0$ for all $a \in A$, and such that $p_{2n} = 1_k$ for all $n \in \mathbb{N}$.

Now, for each n, the restriction $p|_{[2n,2n+1]}$ is uniformly continuous, whence there is some $r_n \in (0,1)$ such that if $t, s \in [2n, 2n+1]$ satisfy $|t - s| \leqslant 1 - r_n$, then $\|p_t - p_s\| < 1/2$. For each $l \in \mathbb{N} \cup \{0\}$, define $p^{(l)} : E \to M_m(\mathcal{L}(F))$ to be the function whose restriction to $[2n, 2n+1]$ is defined by

$$p^{(l)}_t := p_{2n+(t-2n)(r_n)^l}.$$

Fix a sequence $(s_l)_{l=0}^{\infty}$ of isometries as in Remark 2.3 and consider the formal difference

$$x_\infty := \left[\sum_{l=0}^{\infty} s_l p^{(l)} s_l^*\right] - \left[\sum_{l=0}^{\infty} s_l 1_k s_l^*\right]$$

where the sum converges striclty in $M_m(\mathcal{L}(F)) \cong \mathcal{L}(F^{\oplus m})$ pointwise in t (we are abusing notation slightly: we should really have replaced s_l by $1_{M_m(\mathbb{C})} \otimes s_l$). As $r_n < 1$ and as $p_{2n} = 0$ for each n, we see that for any t, $p_t^{(l)} - 1_k \to 0$ as $l \to \infty$; it follows from this and the fact that each s_l commutes with the representation of A that x_∞ gives a well-defined element of $K_0(C^0_{L,c}(\pi; E))$.

Now, let us consider the element $x_\infty + x$ of $K_0(C^0_{L,c}(\pi; E))$. We claim this equals x_∞. As K_0 is a group, this forces $x = 0$, and thus $K_0(C^0_{L,c}(\pi; E)) = 0$ as required. Indeed, first note that conjugating by the isometry

$$s := \sum_{l=0}^{\infty} s_{l+1} s_l^*$$

in the multiplier algebra of $C^0_{L,c}(\pi; E)$ and applying Lemma 3.8 shows that

$$x_\infty = \left[\sum_{l=1}^{\infty} s_l p^{(l-1)} s_l^*\right] - \left[\sum_{l=1}^{\infty} s_l 1_k s_l^*\right].$$

The choice of the sequence (r_n) and Lemma 3.5 guarantees the existence of a homotopy between $p^{(l-1)}$ and $p^{(l)}$ for each $l \geqslant 1$, and moreover that these homotopies can be assumed equicontinuous as l varies. It follows that

$$x_\infty = \left[\sum_{l=1}^{\infty} s_l p^{(l)} s_l^*\right] - \left[\sum_{l=1}^{\infty} s_l 1_k s_l^*\right] \tag{5}$$

On the other hand, applying Lemma 3.8 again, we have that

$$x = [s_0 p s_0^*] - [s_0 1_k s_0^*].$$

Hence combining this with line above (5) and also Lemma 3.7

$$\begin{aligned} x + x_\infty &= [s_0 p s_0^*] - [s_0 1_k s_0^*] + \left[\sum_{l=1}^{\infty} s_l p^{(l)} s_l^*\right] - \left[\sum_{l=1}^{\infty} s_l 1_k s_l^*\right] \\ &= \left[\sum_{l=0}^{\infty} s_l p^{(l)} s_l^*\right] - \left[\sum_{l=0}^{\infty} s_l 1_k s_l^*\right] \\ &= x_\infty \end{aligned}$$

and we are done. □

We finish this section with some technical results that we will need later. The first goal is to show that $K_*(C_{L,c}(\pi))$ only really depends on information 'at $t = \infty$' in some sense. This is made precise in Corollary 3.11 below, but we need some more notation first.

Definition 3.9. Let $\pi : A \to \mathcal{L}(E)$ be a representation of A. Define $I_{L,c}(\pi)$ to be the ideal in $C_{L,c}(\pi)$ consisting of all functions b such that $ab \in C_0([1,\infty),\mathcal{K}(E))$ for all $a \in A$. Define $Q_L(\pi) := C_{L,c}(\pi)/I_{L,c}(\pi)$ to be the corresponding quotient.

Lemma 3.10. *Let $\pi : A \to \mathcal{L}(E)$ be an infinite multiplicity representation of A. Then $I_{L,c}(\pi)$ has trivial K-theory.*

Proof. Set $I_{L,u}(\pi) := C_{L,u}(\pi) \cap I_{L,c}(\pi)$. The same argument in the proof of Theorem 3.4 shows that the inclusion $I_{L,u}(\pi) \to I_{L,c}(\pi)$ induces an isomorphism on K-theory. It thus suffices to prove that $K_*(I_{L,u}(\pi)) = 0$, which we now do.

Let $(s_n)_{n=0}^{\infty}$ be a sequence of isometries in $\mathcal{L}(E)$ that commute with A, and that have orthogonal ranges as in Remark 2.3. We regard each s_n as an isometry in the multiplier algebra of $I_{L,u}(\pi)$ by having it act pointwise in t. Define

$$\iota : I_{L,u}(\pi) \to I_{L,u}(\pi), \quad b \mapsto s_0 b s_0^*,$$

which is a $*$-homomorphism that induces the identity map on K-theory by Lemma 3.8. On the other hand, for each $s \geqslant 0$, define a $*$-endomorphism α_s of $I_{L,u}(\pi)$ by the formula $\alpha_s(b)_t := b_{t+s}$. Note that for each $b \in \mathcal{L}(E)$, the sum

$$\sum_{n=1}^{\infty} s_n b s_n^*$$

converges in the strict topology of $\mathcal{L}(E) = M(\mathcal{K}(E))$. It is therefore not too hard to see that we get a $*$-homomorphism

$$\alpha : I_{L,u}(\pi) \to I_{L,u}(\pi), \quad \alpha(b) := \sum_{n=1}^{\infty} s_n \alpha_n(b) s_n^*$$

(the image is in $I_{L,u}(\pi)$ as $ab_t \to 0$ as $t \to \infty$ for all $a \in A$, which implies that for each fixed t and any $a \in A$, $a\alpha_n(b_t) \to 0$ as $n \to \infty$). Now, the maps α and ι have orthogonal ranges, whence by Lemma 3.7 $\alpha + \iota$ is also a $*$-homomorphism, and we have that as maps on K-theory, $\alpha_* + \iota_* = (\alpha_* + \iota_*)$. Define $s := \sum_{n=0}^{\infty} s_{n+1}s_n^*$ (convergence in the strict topology), which we think of as a multiplier of $I_{L,u}(\pi)$. Applying Lemma 3.8 again, we see that $\iota + \alpha$ induces the same map on K-theory as the map $b \mapsto s(\iota(b) + \alpha(b))s^*$, which is the map

$$I_{L,u}(\pi) \to I_{L,u}(\pi), \quad b \mapsto \sum_{n=1}^{\infty} s_n \alpha_{n-1}(b) s_n^*.$$

On the other hand, using that elements of $I_{L,u}(\pi)$ are uniformly continuous, we get a homotopy

$$b \mapsto \sum_{n=1}^{\infty} s_n \alpha_{n-1+r}(b) s_n^*, \quad r \in [0,1]$$

between this map and α. In other words, we now have that $\alpha_* + \iota_* = \alpha_*$ as maps on K-theory. This forces ι_* to be the zero map on $K_*(I_{L,u}(\pi))$. However, we also observed already that ι_* is the identity map, so $K_*(I_{L,u}(\pi))$ is indeed zero. □

The following corollary is immediate from the six-term exact sequence in K-theory.

Corollary 3.11. *Let $\pi : A \to \mathcal{L}(E)$ be an infinite multiplicity representation of A on a Hilbert B-module. Then the canonical quotient map $C_{L,c}(\pi) \to Q_L(\pi)$ induces an isomorphism on K-theory.* □

We will need one more definition and lemma about the structure of $C_{L,c}(\pi)$.

Definition 3.12. Let $\pi : A \to \mathcal{L}(E)$ be a representation of A on a Hilbert B-module. Define

$$C_{L,c}(\pi; \mathcal{K}) := C_b([1,\infty), \mathcal{K}(E)) \cap C_{L,c}(\pi),$$

which is an ideal in $C_{L,c}(\pi)$.

Lemma 3.13. *Let $\pi : A \to \mathcal{L}(E)$ be a representation of A on a Hilbert B-module. With notation as in Definitions 3.9 and 3.12, we have*

$$C_{L,c}(\pi) = C_{L,c}(\pi; \mathcal{K}) + I_{L,c}(\pi).$$

In particular, the restriction of the quotient map $C_{L,c}(\pi) \to Q_L(\pi)$ to $C_{L,c}(\pi; \mathcal{K})$ is surjective.

Proof. Let (h_n) be a sequential approximate unit for A, and define $h \in C_{ub}([1,\infty), A)$ by setting $h_t := (n+1-t)h_n + (t-n)h_{n+1}$ for $t \in [n, n+1]$. Then a direct check using that $[a, h] \in C_0([1,\infty), A)$ for any $a \in A$ shows that h defines a multiplier of $C_{L,c}(\pi)$. Moreover, for any $b \in C_{L,c}(\pi)$, $b = (1-h)b + hb$, and one checks directly that $(1-h)b$ is in $I_{L,c}(\pi)$ and that hb is in $C_{L,c}(\pi; \mathcal{K})$. This gives the result on the sum, and the result on the quotient follows immediately. □

Our final goal in this section is to check that the isomorphisms from Theorem 3.2 and Theorem 3.4 are compatible with a special case of functoriality for KK-theory.

Let C be a separable C^*-algebra, and let $\phi : B \to C$ be a $*$-homomorphism. Let E be a Hilbert B-module, and let $E \otimes_\phi C$ be the internal tensor product defined using ϕ, which is a Hilbert C-module. As discussed on [15, page 42] there is a canonical $*$-homomorphism

$$\Phi : \mathcal{L}(E) \to \mathcal{L}(E \otimes_\phi C), \quad a \mapsto a \otimes 1_C.$$

Let $\pi_B : A \to \mathcal{L}(E)$ and $\pi_C : A \to \mathcal{L}(F)$ be representations of A on Hilbert B- and C-modules respectively.

Definition 3.14. With notation as above, a *covering isometry* for ϕ (with respect to π_B and π_C) is any isometry $v \in C_b([1,\infty), \mathcal{L}(E \otimes_\phi C, F))$ such that

$$v^* \pi_C(a) v - \Phi \circ \pi_B(a) \in C_0([1,\infty), \mathcal{K}(E \otimes_\phi C))$$

for all $a \in A$.

Lemma 3.15. *With notation as above, if v is a covering isometry for ϕ, then the formula*

$$\phi^v : C_{L,c}(\pi_B) \to C_{L,c}(\pi_C), \quad \phi^v(b)_t := v_t \Phi(b_t) v_t^*$$

gives a well-defined $*$-homomorphism. Moreover, the induced map

$$\phi^v_* : K_*(C_{L,c}(\pi_B)) \to K_*(C_{L,c}(\pi_C))$$

on K-theory does not depend on the choice of v. Finally, if π_C is strongly absorbing, then a covering isometry for ϕ always exists, and can be taken to belong to $C_{ub}([1,\infty), \mathcal{L}(E \otimes_\phi C, F))$ (i.e. to be uniformly continuous, not just continuous).

Proof. Let v be a covering isometry for ϕ. For notational simplicity, write $\sigma := \Phi \circ \pi_B$. Using Lemma 2.7 we have that

$$\pi_C(a)v - v\sigma(a) \in C_0([1,\infty), \mathcal{K}(E \otimes_\phi C, F))$$

for all $a \in A$. Note that for $a \in A$, $b \in C_{L,c}(\pi_B)$

$$\pi_C(a)\phi^v(b) = (\pi_C(a)v - v\sigma(a))(\Phi(b_t))v^* + v\Phi(\pi_B(a)b)v^*;$$

using that Φ takes $\mathcal{K}(E)$ to $\mathcal{K}(E \otimes_\phi C)$ (see [15, Proposition 4.7]), this shows that $\pi_C(a)\phi^v(b) \in C_b([1,\infty), \mathcal{K}(F))$. Similarly,

$$\begin{aligned}[\pi_C(a), \phi^v(b)] = \big(\pi_C(a)v - v\sigma(a)\big)\Phi(b)v^* + v\Phi([\pi(a), b])v^* \\ + v\Phi(b)\big(v^*\pi_C(a) - \sigma(a)v^*\big),\end{aligned}$$

whence $[\pi_C(a), \phi^v(b)] \in C_0([1,\infty), \mathcal{K}(F))$. It follows that ϕ^v is indeed a well-defined $*$-homomorphism $C_{L,c}(\pi_B) \to C_{L,c}(\pi_C)$.

Let now v, w be possibly different covering isometries for ϕ. Using similar computations to the above, one checks that wv^* is an element of the multiplier algebra of $C_{L,c}(\pi_C)$ that conjugates the $*$-homomorphisms ϕ^v and ϕ^w to each other. The fact that $\phi^v_* = \phi^w_*$ as maps $K_*(C_{L,c}(\pi_B)) \to K_*(C_{L,c}(\pi_C))$ follows from this and Lemma 3.8.

Finally, if π_C is strongly absorbing, then covering isometries exist, and can be assumed uniformly continuous, by Proposition 2.6. ∎

Definition 3.16. Let $\pi_B : A \to \mathcal{L}(E)$ and $\pi_C : A \to \mathcal{L}(F)$ be representations of A on a Hilbert B-module and Hilbert C-module respectively, with π_C strongly absorbing. Let $\phi : B \to C$ be any $*$-homomorphism. Then Lemma 3.15 gives a well-defined homomorphism $K_*(C_{L,c}(\pi_B)) \to K_*(C_{L,c}(\pi_C))$, which we denote ϕ_*.

On the other hand, for a $*$-homomorphism $\phi : B \to C$, let us write $\phi_* : KK(A, B) \to KK(A, C)$ for the usual functorially induced map on KK-theory. The following lemma gives compatibility between these two maps.

Lemma 3.17. *With notation as above, assume that both π_B and π_C are strongly absorbing, and let $KK(A, B) \to K_0(C_{L,c}(\pi_B))$ and $KK(A, C) \to K_0(C_{L,c}(\pi_C))$ be the isomorphisms from Theorem 3.2 and Theorem 3.4. Then the diagram*

$$\begin{tikzcd} KK(A,B) \arrow[r] \arrow[d, "\phi_*"] & K_0(C_{L,c}(\pi_B)) \arrow[d, "\phi_*"] \\ KK(A,C) \arrow[r] & K_0(C_{L,c}(\pi_C)) \end{tikzcd}$$

commutes.

Proof. The proof is unfortunately long as there is a lot to check, but the checks are fairly routine. We recall first the precise form of the isomorphism $KK_*(A, B) \to K_*(C_{L,c}(\pi_B))$ of Theorem 3.2. It is a composition of the following maps (see also [8, Definition 3.1] for the various algebras involved).

(i) The Paschke duality isomorphism $P : KK(A, B) \to K_1(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B))$ of [25, Theorem 3.2], where $\mathcal{D}(\pi_B) := \{b \in \mathcal{L}(E) \mid [b, a] \in \mathcal{K}(E) \text{ for all } a \in A\}$, and $\mathcal{C}(\pi_B) := \{b \in \mathcal{D}(\pi_B) \mid ab \in \mathcal{K}(E) \text{ for all } a \in A\}$.

(ii) The map on K-theory $\iota_* : K_1(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)) \to K_1(\mathcal{D}_T(\pi_B)/\mathcal{C}_T(\pi_B))$ induced by the constant inclusion $\iota : \mathcal{D}(\pi_B) \to \mathcal{D}_T(\pi_B)$, where $\mathcal{D}_T(\pi_B) := C_{ub}([1, \infty), \mathcal{D}(\pi_B))$ and $\mathcal{C}_T(\pi_B) := C_{ub}([1, \infty), \mathcal{C}(\pi_B))$.

(iii) The map on K-theory $\eta_*^{-1} : K_1(\mathcal{D}_T(\pi_B)/\mathcal{C}_T(\pi_B)) \to K_1(\mathcal{D}_L(\pi_B)/C_{L,u}(\pi_B))$ which is induced by the inverse (it turns out to be an isomorphism of C^*-algebras) of the map $\eta : \mathcal{D}_T(\pi_B)/\mathcal{C}_T(\pi_B) \to \mathcal{D}_L(\pi_B)/C_{L,u}(\pi_B)$ induced by the inclusion $\mathcal{D}_L(\pi_B) \to \mathcal{D}_T(\pi_B)$, where $\mathcal{D}_L(\pi_B) := \{b \in \mathcal{D}_T(\pi_B) \mid [a, b_t] \to 0 \text{ as } t \to \infty \text{ for all } a \in A\}$.

(iv) The usual K-theory boundary map $\partial : K_1(\mathcal{D}_L(\pi_B)/C_{L,u}(\pi_B)) \to K_0(C_{L,u}(\pi_B))$.

(v) The isomorphism $\kappa_* : K_0(C_{L,u}(\pi_B)) \to K_0(C_{L,c}(\pi_B))$ of Theorem 3.4 induced by the canonical inclusion.

Now, if v is a uniformly continuous covering isometry for ϕ, then one sees from analogous arguments to those given in the proof of Lemma 3.15 that the formula

$$\phi^v(b)_t := v_t \Phi(b_t) v_t^*$$

from Lemma 3.15 also defines $*$-homomorphisms

$$\phi^v : \left\{ \begin{array}{l} \mathcal{D}_L(\pi_B)/C_{L,u}(\pi_B) \to \mathcal{D}_L(\pi_C)/C_{L,u}(\pi_C) \\ \mathcal{D}_T(\pi_B)/\mathcal{C}_T(\pi_B) \to \mathcal{D}_T(\pi_C)/\mathcal{C}_T(\pi_C) \end{array} \right\}.$$

Moreover, the formula

$$\phi^{v_1}(b) := v_1 \Phi(b) v_1^*$$

defines a $*$-homomorphism $\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B) \to \mathcal{D}(\pi_C)/\mathcal{C}(\pi_B)$. Putting all this together, we get a diagram

$$\begin{tikzcd} K_1(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)) \arrow[r, "\iota_*"] \arrow[d, "\phi_*^{v_1}"] & K_1(\mathcal{D}_T(\pi_B)/\mathcal{C}_T(\pi_B)) \arrow[r, "\eta_*^{-1}"] \arrow[d, "\phi_*^{v}"] & K_1(\mathcal{D}_L(\pi_B)/C_{L,u}(\pi_B)) \arrow[r, "\partial"] \arrow[d, "\phi_*^{v}"] & K_0(C_{L,c}(\pi_B)) \arrow[r, "\kappa_*"] \arrow[d, "\phi_*^{v}"] & K_0(C_{L,c}(\pi_B)) \arrow[d, "\phi_*^{v}"] \\ K_1(\mathcal{D}(\pi_C)/\mathcal{C}(\pi_C)) \arrow[r, "\iota_*"] & K_1(\mathcal{D}_T(\pi_C)/\mathcal{C}_T(\pi_C)) \arrow[r, "\eta_*^{-1}"] & K_1(\mathcal{D}_L(\pi_C)/C_{L,u}(\pi_C)) \arrow[r, "\partial"] & K_0(C_{L,c}(\pi_C)) \arrow[r, "\kappa_*"] & K_0(C_{L,c}(\pi_C)) \end{tikzcd} \tag{6}$$

We claim that this commutes. Indeed, the first square commutes as ι_* is an isomorphism on K-theory ([8, Proposition 4.3 (b)]), whence its inverse on the level of K-theory is the map induced by the evaluation-at-one homomorphism $e : \mathcal{D}_T(\pi_B) \to \mathcal{D}(\pi_B)$, and the diagram

$$\begin{tikzcd} K_1(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)) \arrow[d, "\phi_*^{v_1}"] & K_1(\mathcal{D}_T(\pi_B)/\mathcal{C}_T(\pi_B)) \arrow[l, "e_*"'] \arrow[d, "\phi_*^{v}"] \\ K_1(\mathcal{D}(\pi_C)/\mathcal{C}(\pi_C)) & K_1(\mathcal{D}_T(\pi_C)/\mathcal{C}_T(\pi_C)) \arrow[l, "e_*"'] \end{tikzcd}$$

commutes on the level of $*$-homomorphisms. The second square in line (6) commutes as the diagram

$$\begin{tikzcd} K_1(\mathcal{D}_T(\pi_B)/\mathcal{C}_T(\pi_B)) \arrow[d, "\phi_*^v"] & K_1(\mathcal{D}_L(\pi_B)/C_{L,u}(\pi_B)) \arrow[l, "\eta_*"'] \arrow[d, "\phi_*^v"] \\ K_1(\mathcal{D}_T(\pi_C)/\mathcal{C}_T(\pi_C)) & K_1(\mathcal{D}_L(\pi_C)/C_{L,u}(\pi_C)) \arrow[l, "\eta_*"'] \end{tikzcd}$$

commutes on the level of $*$-homomorphisms. The third square commutes by naturality of the boundary map in K-theory. Finally, the fourth square commutes as it commutes on the level of $*$-homomorphisms.

Now, the diagram in the statement in the lemma 'factors' as the rectangle from line (6), augmented on the left with the diagram below

$$\begin{tikzcd} KK(A,B) \arrow[r, "P"] \arrow[d, "\phi_*"] & K_1(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)) \arrow[d, "\phi_*^{v_1}"] \\ KK(A,C) \arrow[r, "P"] & K_1(\mathcal{D}(\pi_C)/\mathcal{C}(\pi_C)) \end{tikzcd} \tag{7}$$

involving the Paschke duality isomorphism. To complete the proof, it suffices to show that this commutes.

For this, we use the ungraded picture of KK-theory, so a cycle for $KK(A,B)$ consists of a triple (σ, G, w), where $\sigma : A \to \mathcal{L}(G)$ is a representation, and $u \in \mathcal{L}(E)$ is such that $a(uu^* - 1)$, $a(u^*u - 1)$, and $[a,u]$ are all in $\mathcal{K}(G)$ for all $a \in A$. In this picture, the Paschke duality isomorphism (see [25, Section 3] and [24, Remarque 2.8]) can be described as follows. Take a cycle (σ, G, w) for $KK(A,B)$. Adding the degenerate cycle $(\pi_B, E, 1)$ gives an equivalent cycle $(\sigma \oplus \pi_B, G \oplus E, w \oplus 1)$. As in [25, Theorem 2.5, condition (3)] we may use that π_B is absorbing to find a unitary $U \in \mathcal{L}(G \oplus E, E)$ such that $U(\sigma(a) \oplus \pi_B(a))U^* - \pi_B(a) \in \mathcal{K}(E)$ for all $a \in A$. We then get a unitarily equivalent cycle

$$(U(\sigma \oplus \pi_B)U^*, E, U(w \oplus 1)U^*)$$

for $KK(A,B)$. The element $U(w \oplus 1)U^*$ is then unitary in $\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)$ and we define $P[\sigma, G, w]$ to be the class $[U(w \oplus 1)U^*]$. This construction induces the Paschke duality isomorphism.

Now, to keep notation under control, let us start with an element of $KK(A,B)$ represented by a cycle of the form (π_B, E, w) (such representatives always exist). Then the 'right-down' composition

$$\begin{tikzcd} KK(A,B) \arrow[r, "P"] & K_1(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)) \arrow[d, "\phi_*^{v_1}"] \\ & K_1(\mathcal{D}(\pi_C)/\mathcal{C}(\pi_C)) \end{tikzcd}$$

from line (7) takes $[\pi_B, E, w]$ first to $[w]$, and then to $[v_1(w \otimes 1_C)v_1^* + (1 - v_v v_1^*)]$. On the other hand, the 'down-right' composition

$$\begin{tikzcd} KK(A,B) \arrow[d, "\phi_*"] & \\ KK(A,C) \arrow[r, "P"] & K_1(\mathcal{D}(\pi_C)/\mathcal{C}(\pi_C)) \end{tikzcd}$$

from line (7) takes $[\pi_B, E, w]$ first to $[\pi_B \otimes 1_C, E \otimes C, w \otimes 1_C]$, and then to $[U((w \otimes 1_C) \oplus 1)U^*]$, where $U \in \mathcal{L}((E \otimes C) \oplus F, F)$ is a unitary such that $U((\pi_B(a) \otimes 1_C) \oplus \pi_C(a))U^* - \pi_C(a) \in \mathcal{K}(F)$ for all $a \in A$. Our task is therefore to establish the identity

$$[U((w \otimes 1_C) \oplus 1)U^*] = [v_1(w \otimes 1_C)v_1^* + (1 - v_1 v_1^*)] \tag{8}$$

in $K_1(\mathcal{D}(\pi_C)/\mathcal{C}(\pi_C))$.

Now, as (π_C, F) is strongly absorbing, it is equivalent to $(\pi_C \oplus \pi_C, F \oplus F)$. We may assume that the original isometry $v \in C_{ub}([1,\infty), \mathcal{L}(E \otimes C, F))$ takes values in the first summand above. It then follows that if $s : F \to F \oplus F$ is the isometric inclusion as the second summand that we have that s takes image in $(1 - v_1 v_1^*)F$, and that $s^*(\pi_C(a) \oplus \pi_C(a))s = \pi_C(a)$ for all $a \in A$. One then checks that $V := (s \oplus v_1)U^* \in LL(E)$ defines a multiplier of $\mathcal{D}(\pi_C)$ (and therefore of $\mathcal{D}(\pi_C)/\mathcal{C}(\pi_C)$). We compute that

$$V(U((w \otimes 1_C) \oplus 1)U^*)V^* + (1 - VV^*) = v_1(w \otimes 1_C)v_1^* + (1 - v_1 v_1^*).$$

On the other hand, the $*$-homomorphism $\mathrm{ad}_V : \mathcal{D}(\pi_C)/\mathcal{C}(\pi_C) \to \mathcal{D}(\pi_C)/\mathcal{C}(\pi_C)$ induces the identity on K-theory by Lemma 3.8, and is concretely given by the formula

$$K_1(\mathcal{D}(\pi_C)/\mathcal{C}(\pi_C)) \to K_1(\mathcal{D}(\pi_C)/\mathcal{C}(\pi_C)), \quad [u] \mapsto [VuV^* + (1 - VV^*)].$$

We have thus established the identity in line (8), which completes the proof.

□

4 Paths of projections

Our goal in this section is to introduce a new model of KK-theory based on paths of projections. Throughout this section, A and B are separable C^*-algebras.

We will need some more terminology about representations.

Definition 4.1. Let $\pi : A \to \mathcal{L}(E)$ e a representation of A on a Hilbert B-module.

- π is *graded* if it comes with a fixed decomposition $(\pi, E) = (\pi_0 \oplus \pi_1, E_0 \oplus E_1)$ as a direct sum of two subrepresentations. If π is graded, the *neutral projection* is the projection $e \in \mathcal{L}(E)$ onto the first summand in $E = E_0 \oplus E_1$.

- π is *substantial* if it is graded, if $(\pi_0, E_0) = (\pi_1, E_1)$ in the given decomposition, and if (π_0, E_0) comes with a fixed decomposition $(\pi_0, E_0) = (\sigma^\infty, F^\infty)$ as an infinite amplification of another representation.

- π is *substantially absorbing* if it is substantial, and if in addition (π_0, E_0) is strongly absorbing.

Note that a representation (π, E) is substantial if and only if

$$(\pi, E) = (1_{\mathbb{C}^2 \otimes \ell^2 \otimes \ell^2} \otimes \sigma, \mathbb{C}^2 \otimes \ell^2 \otimes \ell^2 \otimes F) \tag{9}$$

(a tensor factor of ℓ^2 comes from the infinite multiplicity assumption, and we use an identification $\ell^2 = \ell^2 \otimes \ell^2$ to split off an extra tensorial factor of ℓ^2). We record some useful observations arising from this as a lemma.

Lemma 4.2. *Let $\pi : A \to \mathcal{L}(E)$ be a substantial representation. Arising from a decomposition as in line (9), there are unital inclusions*

$$M_2(\mathbb{C}) \subseteq \mathcal{L}(E) \quad \textit{and} \quad \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$$

as the the C^*-subalgebras

$$M_2(\mathbb{C}) \otimes 1_{\ell^2 \otimes \ell^2 \otimes F} \quad \textit{and} \quad 1_{\mathbb{C}^2} \otimes \mathcal{B}(\ell^2) \otimes 1_{\ell^2 \otimes F}$$

respectively. These inclusions have the following properties:

- *The neutral projection corresponds to the element $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$.*
- *The subalgebras $\mathcal{B}(\ell^2)$ and $M_2(\mathbb{C})$ of $\mathcal{L}(E)$ commute with each other, and with A.*
- *The compositions*

  $$\mathcal{B}(\ell^2) \to \mathcal{L}(E) \to \mathcal{L}(E)/\mathcal{K}(E) \quad \textit{and} \quad M_2(\mathbb{C}) \to \mathcal{L}(E) \to \mathcal{L}(E)/\mathcal{K}(E)$$

  of these inclusions with the quotient map to the Calkin algebra are still injective. □

The following is the key definition of this section.

Definition 4.3. Let $\pi : A \to \mathcal{L}(E)$ be a graded representation, and define $\mathcal{P}^\pi(A, B)$ to be the set of self-adjoint contractions $p \in C_b([1, \infty), \mathcal{L}(E))$ such that:

(i) $p - e \in C_b([1, \infty), \mathcal{K}(E))$[16];

(ii) for all $a \in A$, $[a, p] \in C_0([1, \infty), \mathcal{L}(E))$;

(iii) for all $a \in A$, $a(p^2 - p) \in C_0([1, \infty), \mathcal{K}(E))$.

[16]To make sense of this, we follow our usual conventions and identify e with a constant function in $C_b([1, \infty), \mathcal{L}(E))$.

We will sometimes drop the superscript "$^{\pi}$" and just write "$\mathcal{P}(A,B)$" when it seems unlikely to cause confusion.

Our next goal is to define an equivalence relation on $\mathcal{P}^{\pi}(A,B)$ such that the equivalence classes give a realization of $KK(A,B)$. For this (and other purposes later), it will be convenient to introduce a parameter space Y. Let then $C = C_0(Y)$ be a separable commutative C^*-algebra: for our applications, Y will be one of the intervals $[0,1]$ or $(0,1)$, or the one-point compactification $\overline{\mathbb{N}}$ of the natural numbers. Let (π, E) be a representation of A on a Hilbert B-module, and let $C \otimes E$ denote the tensor product Hilbert $C \otimes B$-module. Let $1 \otimes \pi : A \to \mathcal{L}(C \otimes E)$ be the amplification of π. If π is graded then $1 \otimes \pi$ inherits a grading in a natural way, and so if we are in the graded case we may consider $\mathcal{P}^{1\otimes\pi}(A, C \otimes B)$.

The following lemma characterizes elements of $\mathcal{P}^{1\otimes\pi}(A, C\otimes B)$ in terms of doubly parametrized families $(p_t^y)_{t\in[1,\infty),y\in Y}$.

Lemma 4.4. *Let (π, E) be a graded representation of A on a Hilbert B-module. With notation as above, there is a natural identification between elements p of $\mathcal{P}^{1\otimes\pi}(A, C\otimes B)$ and doubly parametrized families of self-adjoint contractions $(p_t^y)_{t\in[1,\infty),y\in Y}$ that define a function*

$$p : [1,\infty) \to C_b(Y, \mathcal{L}(E)), \quad t \mapsto (y \mapsto p_t^y)$$

with the following properties:

(i) the function $p - e$ is in $C_b([1,\infty), C_0(Y, \mathcal{K}(E)))$;

(ii) $[p, a] \in C_0([1,\infty), C_b(Y, \mathcal{L}(E)))$ for all $a \in A$;

(iii) $a(p^2 - p) \in C_0([1,\infty), C_0(Y, \mathcal{K}(E)))$ for all $a \in A$.

Proof. An element of $\mathcal{P}^{1\otimes\pi}(A, C \otimes B)$ is a function $p : [1,\infty) \to \mathcal{L}(C \otimes E)$ satisfying the conditions of Definition 4.3. Using the canonical identifications

$$\mathcal{K}(C \otimes E) = C \otimes \mathcal{K}(E) = C_0(Y, \mathcal{K}(E))$$

and the fact that $p - e \in C_b([1,\infty), \mathcal{K}(E))$, we identify p with a function $p : [1,\infty) \to C_b(Y, \mathcal{L}(E))$ (with image in the subset $C_0(Y, \mathcal{K}(E)) + \{e\} \subseteq C_b(Y, \mathcal{L}(E))$). The remaining checks are direct. $\square$

Definition 4.5. Elements p^0 and p^1 of $\mathcal{P}^\pi(A,B)$ are *homotopic* if (with notation as in Lemma 4.4) there is an element $p = (p_t^s)_{t\in[1,\infty),s\in[0,1]}$ in $\mathcal{P}^{1\otimes\pi}(A, C[0,1]\otimes B)$ that agrees with p^0 and p^1 at the endpoints. We write $p^0 \sim p^1$ if p^0 and p^1 are homotopic, and write $KK_{\mathcal{P}}^\pi(A,B)$ for the quotient set $\mathcal{P}^\pi(A,B)/\sim$.

We will need the following elementary lemma a few times, so record it here.

Lemma 4.6. *Say p and q are elements of $\mathcal{P}^\pi(A,B)$ such that $p_t - q_t \to 0$ as $t \to \infty$. Then $p \sim q$.*

Proof. A straight line homotopy $(sp + (1-s)q)_{s\in[0,1]}$ works: we leave the direct checks involved to the reader. □

In order to define a semi-group structure on $KK_{\mathcal{P}}^\pi(A,B)$, we assume π is substantial as in Definition 4.1, and fix a tensorial decomposition as in line (9) (which will remain fixed for the rest of the section). Fix also two isometries s_1 and s_2 in $\mathcal{B}(\ell^2)$ that satisfy the Cuntz relation $s_1s_1^* + s_2s_2^* = 1$. Using the canonical (unital) inclusion $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ from Lemma 4.2, we think of these isometries as adjointable operators on E that commute with $A \subseteq \mathcal{L}(E)$ and with the neutral projection $e \in \mathcal{L}(E)$.

Lemma 4.7. *Let $\pi : A \to \mathcal{L}(E)$ be a substantial representation. Then with notation as above, the operation defined by*

$$[p] + [q] := [s_1ps_1^* + s_2qs_2^*]$$

makes $KK_{\mathcal{P}}^\pi(A,B)$ into an abelian semigroup. The operation does not depend on the choice of s_1, s_2 within $\mathcal{B}(\ell^2)$.

Proof. As the unitary group of $\mathcal{B}(\ell^2)$ is connected (in the norm topology), conjugation by a unitary in $\mathcal{B}(\ell^2)$ induces the trivial map on $KK_{\mathcal{P}}^\pi(A,B)$. Hence conjugating by the unitaries $s_1s_2^* + s_2s_1^*$ and $s_1s_1s_1^* + s_1s_2s_1^*s_2^* + s_2s_2^*s_2^*$ show that the operation is commutative and associative. On the other hand, if $t_1, t_2 \in \mathcal{B}(\ell^2)$ also satisfy the Cuntz relation, then conjugating by the unitary $s_1t_1^* + s_2t_2^*$ shows that the pairs (s_1, s_2) and (t_1, t_2) induce the same operation on $KK_{\mathcal{P}}^\pi(A,B)$. □

Our next goal is to show that the semi-group $KK^{\pi}_{\mathcal{P}}(A,B)$ is a monoid. We first state a well-known lemma about paths of projections in a C^*-algebra. It follows from the arguments of [11, Proposition 4.1.7 and Corollary 4.1.8], for example.

Lemma 4.8. *Let I be either $[a,b]$ or $[a,\infty)$ for some $a,b \in \mathbb{R}$, and let $(p_t)_{t\in I}$ be a continuous path of projections in a C^*-algebra D. Then there is a continuous path of unitaries $(u_t)_{t\in I}$ in D such that $u_a = 1$, and such that $p_t = u_t p_a u_t^*$ for all t.* □

Lemma 4.9. *Let $\pi : A \to \mathcal{L}(E)$ be a substantial representation of A. Let p be an element of $\mathcal{P}^{\pi}(A,B)$, and let v be an isometry in the canonical copy of $\mathcal{B}(\ell^2) \subseteq \mathcal{L}$ from Lemma 4.2. Then the element*

$$q := vpv^* + (1 - vv^*)e \in C_{ub}([1,\infty),\mathcal{L}(E))$$

is in $\mathcal{P}^{\pi}(A,B)$ and satisfies $p \sim q$.

Proof. For each $n \geqslant 1$, a compactness argument gives a finite rank projection

$$e_n \in \mathcal{K}(\ell^2) \subseteq \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$$

(where the inclusion is that from Lemma 4.2) such that

$$\|(1-e_n)(p_t - e)\| < \frac{1}{n}$$

for all $t \in [1, n+1]$. Choose now a projection $r_1 \geqslant e_1$ such that $r_1 - e_1$ and $1 - r_1$ both have infinite rank. Given r_n, define r_{n+1} to be the max of r_n and e_{n+1}. In this way we get an increasing sequence $r_1 \leqslant r_2 \leqslant \cdots$ of projections in $\mathcal{B}(\ell^2)$ such that $r_n \geqslant e_m$ for all n and all $m \leqslant n$, and such that $r_n - e_m$ and $1 - r_n$ both have infinite rank for all n and all $m \leqslant n$. For each n, $(1-e_n)r_n$ and $(1-e_n)r_{n+1}$ are projections with the same dimensional kernel and image as operators on $(1-e_n)\ell^2$, and are thus connected by a continuous path of projections $(r^0_t)_{t\in[n,n+1]}$ in $\mathcal{B}((1-e_n)\ell^2)$. Set $r_t := e_n + r^0_t$ for $t \in [n, n+1]$. In this way we get a continuous path of projections $r = (r_t)_{t\in[1,\infty)}$ in $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ such that if $\lfloor t \rfloor$ is the floor function of t then

$$\|(1-r_t)(p_t - e)\| \leqslant \|(1-e_{\lfloor t \rfloor})(p_t - e)\| < \frac{1}{\lfloor t \rfloor}, \tag{10}$$

and such that r_t and $1 - r_t$ have infinite rank as operators on ℓ^2 for each t.

Note now that as r_t commutes with e, line (10) implies in particular that $\|[r_t, p_t]\| < 2/\lfloor t \rfloor$. Define $p' \in C_{ub}([1,\infty), \mathcal{L}(E))$ by

$$p'_t := r_t p_t r_t + (1 - r_t)e.$$

As $r_t p_t r_t - r_t e$ is in $\mathcal{K}(E)$ for all t, we see that $p'_t - e$ is in $\mathcal{K}(E)$ for all t. Moreover,

$$\|p'_t - p_t\| \leqslant \|[r_t, p_t]\| + \|(1 - r_t)(p_t - e)\| \to 0 \quad \text{as} \quad t \to \infty,$$

and so we that $p' := (p'_t)$ defines an element of $\mathcal{P}^\pi(A, B)$ and that $p' \sim p$ by Lemma 4.6.

Now, let $v \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ be an isometry as in the original statement. Lemma 4.8 gives a continuous path $(u^r_t)_{t \in [1,\infty)}$ of unitaries in $\mathcal{B}(\ell^2)$ such that $r_t = u^r_t r_1 (u^r_t)^*$ for all t. Similarly, we get a continuous path of unitaries $(u^v_t)_{t \in [1,\infty)}$ such that $u^v_t(1 - vv^* + v(1 - r_1)v^*)(u^v_t)^* = 1 - vv^* + v(1 - r_t)v^*$ for all t. Choose any partial isometry $w \in \mathcal{B}(\ell^2)$ such that $ww^* = r_1$ and $w^*w = 1 - vv^* + v(1 - r_1)v^*$ (such exists as r_1 and $1 - vv^* + v(1 - r_1)v^*$ are both infinite rank), and define $w_t := u^r_t w (u^v_t)^*$. Then $(w_t)_{t \in [1,\infty)}$ is a continuous path of partial isometries in $\mathcal{B}(\ell^2)$ such that $w_t w_t^* = 1 - r_t$ and $w_t^* w_t = 1 - vv^* + v(1 - r_t)v^*$. Define

$$u_t := v r_t + w_t^* \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E).$$

Then $u = (u_t)_{t \in [1,\infty)}$ is a continuous path of unitaries such that $up'u^* = vp'v^* + (1 - vv^*)e$. Let $(h^s : \mathcal{U}(\ell^2) \to \mathcal{U}(\ell^2))_{s \in [0,1]}$ be a norm-continuous contraction of the unitary group of ℓ^2 to the identity element (such exists by Kuiper's theorem: see for example [4, Theorem on page 433]) and note that the path $(h^s(u)p'h^s(u^*))_{s \in [0,1]}$ shows that $p' \sim vp'v^* + (1 - vv^*)e$. In conclusion, we have that

$$p \sim p' \sim vp'v^* + (1 - vv^*)e \sim vpv^* + (1 - vv^*)e$$

and are done. □

Corollary 4.10. *Let (π, E) be a substantial representation of A. Then for any $p \in \mathcal{P}^{\pi}(A, B)$, we have $s_1 p s_1^* + s_2 e s_2^* \sim p$. In particular, the semigroup $KK^{\pi}_{\mathcal{P}}(A, B)$ is a commutative monoid with identity given by the class $[e]$ of the neutral projection.*

Proof. Apply Lemma 4.9 with $v = s_1$, whence $1 - vv^* = s_2 s_2^*$, and use that s_2 commutes with e. $\square$

Our next goal, which is the main point of this section, is to show that if π is in addition substantially absorbing then $KK^{\pi}_{\mathcal{P}}(A, B) \cong KK(A, B)$ (and therefore in particular that $KK^{\pi}_{\mathcal{P}}(A, B)$ is a group). We need some preliminaries.

Let (π, E) be a substantial representation of A, and keep the fixed decomposition of line (9) and the Cuntz isometries of Lemma 4.7. Lemma 3.13 gives us a surjection $\rho : C_{L,c}(\pi; \mathcal{K}) \to Q_L(\pi)$. This induces a $*$-homomorphism $\overline{\rho} : M(C_{L,c}(\pi; \mathcal{K})) \to M(Q_L(\pi))$ on multiplier algebras, which is uniquely determined by the condition that $\overline{\rho}(m) \cdot \rho(b) = \rho(mb)$ for all $m \in M(C_{L,c}(\pi; \mathcal{K}))$ and $b \in C_{L,c}(\pi; \mathcal{K})$ (see [15, Chapter 2] for this). We define

$$M := \overline{\rho}(M(C_{L,c}(\pi; \mathcal{K}))), \tag{11}$$

which is a unital C^*-subalgebra[17] of $M(Q_L(\pi))$ containing $Q_L(\pi)$ as an ideal.

Lemma 4.11. *With notation as in line (11) above, M has trivial K-theory.*

Proof. The unital inclusion $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ of Lemma 4.2 induces a unital inclusion $\mathcal{B}(\ell^2) \subseteq M(C_{L,c}(\pi; \mathcal{K}))$ by having $\mathcal{B}(\ell^2))$ act pointwise in the variable t (this uses that $\mathcal{B}(\ell^2)$ commutes with A). This in turn descends to a unital inclusion $\mathcal{B}(\ell^2) \subseteq M$. Let $(s_n)_{n=0}^{\infty}$ be a sequence of isometries in $\mathcal{B}(\ell^2) \subseteq M$ with orthogonal ranges.

Consider the maps

$$\iota_0 : M(C_{L,c}(\pi; \mathcal{K})) \to M(C_{L,c}(\pi; \mathcal{K})), \quad b \mapsto s_0 b s_0^*,$$

[17] It could be all of $M(Q_L(\pi))$, although this does not seem to be obvious: note that the noncommutative Tietze extension theorem [15, Proposition 6.8] is not available here as $C_{L,c}(\pi; \mathcal{K})$ is not σ-unital.

and

$$\alpha_0 : M(C_{L,c}(\pi;\mathcal{K})) \to M(C_{L,c}(\pi;\mathcal{K})), \quad b \mapsto \sum_{n=1}^{\infty} s_n b s_n^*$$

(the sum converges in the strict topology of $\mathcal{L}(E)$, pointwise in t). The kernel of the map $\overline{\rho} : M(C_{L,c}(\pi;\mathcal{K})) \to M$ is

$$\{m \in M(C_{L,c}(\pi;\mathcal{K})) \mid mb \in \mathcal{I}_L(\pi) \text{ for all } b \in C_{L,c}(\pi;\mathcal{K})\},$$

whence ι_0 and α_0 descend to well-defined $*$-homomorphisms $\iota, \alpha : M \to M$.

As α and ι have orthogonal ranges, Lemma 3.7 implies that $\alpha + \iota$ is a $*$-homomorphism and that as maps on K-theory, $\alpha_* + \iota_* = (\alpha + \iota)_*$. Moreover, conjugating by the isometry $s := \sum_{n=0}^{\infty} s_n s_{n+1}^* \in \mathcal{B}(\ell^2) \subseteq M$ (the sum converges in the strong topology of $\mathcal{B}(\ell^2)$) and applying Lemma 3.8 implies that $(\alpha + \iota)_* = \alpha_*$ as maps on K-theory. We thus have

$$\alpha_* + \iota_* = (\alpha + \iota)_* = \alpha_*,$$

whence $\iota_* = 0$. However, ι_* is an isomorphism by Lemma 3.8 again, whence $K_*(M)$ is zero as required. □

We need one more preliminary definition and lemma before we get to the isomorphism $KK^{\pi}_{\mathcal{P}}(A, B) \cong KK(A, B)$.

Definition 4.12. For an ideal I in a C^*-algebra N, the *double* of I along N is the C^*-algebra defined by

$$D_N(I) := \{(a, b) \in N \oplus N \mid a - b \in I\}.$$

Note that $D_N(I)$ fits into a short exact sequence

$$0 \longrightarrow I \longrightarrow D_N(I) \longrightarrow N \longrightarrow 0 \tag{12}$$

with arrows $I \to N$ and $D_N(I) \to N$ given by $a \mapsto (a, 0)$ and $(a, b) \mapsto b$ respectively.

Lemma 4.13. *Say I is an ideal in a C^*-algebra N, let $D_N(I)$ be the double from Definition 4.12, and assume that $K_*(N) = 0$. Then $D_N(I)$ has the following properties:*

(i) *The inclusion $I \to D_N(I)$ from line (12) induces an isomorphism on K-theory;*

(ii) *any class in $K_0(D_N(I))$ of the form $[p,p]$ for some projection $p \in M_n(N)$ is zero;*

(iii) *for any $[p,q] \in K_0(D_N(I))$, we have $-[p,q] = [q,p]$;*

(iv) *any element in $K_0(D_N(I))$ can be written as $[p,q]$ for some projection (p,q) in some matrix algebra $M_n(D_N(I))$.*

Proof. Part (i) is immediate from the six-term exact sequence in K-theory. Part (ii) follows as any such class is in the image of the map induced on K-theory by the $*$-homomorphism

$$N \to D_N(I), \quad a \mapsto (a,a)$$

and is thus zero as $K_*(N) = 0$. For part (iii), say $[p,q] \in K_0(D_N(I))$ with $p,q \in M_n(N)$. Then $[p,q] + [q,p] = [p \oplus q, q \oplus p]$. As $p - q \in M_n(I)$, the formula

$$[0,\pi/2] \ni s \mapsto \left(\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{pmatrix} \right)$$

defines a homotopy between $(p \oplus q, q \oplus p)$ and $(p \oplus q, p \oplus q)$ passing through projections in $M_{2n}(D_N(I))$. The latter defines the zero class in K_0 by part (ii), which gives part (iii). Part (iv) follows directly from part (iii). $\square$

Here is the main result of this section.

Theorem 4.14. *Let $\pi : A \to \mathcal{L}(E)$ be a substantially absorbing representation on a Hilbert B-module E. Let M be as in line (11), $Q_L(\pi)$ as in Definition 3.9 and $D_M(Q_L(\pi))$ be as in Definition 4.12. Then the formula*

$$KK^\pi_{\mathcal{P}}(A,B) \to K_0(D_M(Q_L(\pi))), \quad [p] \mapsto [p,e]$$

defines an isomorphism of commutative monoids. In particular $KK^\pi_{\mathcal{P}}(A,B)$ is an abelian group.

Moreover, there is a canonical isomorphism $KK^\pi_{\mathcal{P}}(A,B) \cong KK(A,B)$.

Proof. We first have to show that the map above is well-defined. It is not difficult to see that if $p \in \mathcal{P}^\pi(A, B)$, then (p, e) is a projection in $D_M(Q_L(\pi))$. For well-definedness, we need to show that if $p^0 \sim p^1$ in $\mathcal{P}^\pi(A, B)$, then the projections (p^0, e) and (p^1, e) in $D_M(Q_L(\pi))$ define the same K-theory class. Let then $(p^s)_{s\in[0,1]}$ be a homotopy implementing the equivalence between p^0 and p^1. Let

$$1 \otimes \pi : A \to \mathcal{L}(C[0,1] \otimes E)$$

be the amplification of π to the $C[0,1] \otimes B$-module $C[0,1] \otimes E$, and let $C_{L,c}(1 \otimes \pi)$ be the associated localization algebra. Note that $p := (p^s)_{s\in[0,1]}$ defines an element of the multiplier algebra $M(C_{L,c}(1 \otimes \pi))$ such that $p - e$ is in $C_{L,c}(1 \otimes \pi)$, and so that $[p, e]$ is a well-defined class in $D_{M_C}(Q_L(1 \otimes \pi))$, where M_C is defined analogously to M, but starting with $1 \otimes \pi$.

As E is (substantially) absorbing, Remark A.16 implies that it is isomorphic as a Hilbert B-module to $\ell^2 \otimes B$. Hence we may apply Proposition 2.9 to conclude that $1 \otimes \pi$ is substantially absorbing, and thus there is an isomorphism

$$KK(A, C[0,1] \otimes B) \xrightarrow{\cong} K_0(C_{L,c}(1 \otimes \pi)).$$

Let $\epsilon^0, \epsilon^1 : C[0,1] \otimes B \to B$ be given by evaluation at the endpoints. Lemma 3.17 then gives a commutative diagram

$$\begin{tikzcd} KK(A, C[0,1] \otimes B) \arrow[r, "\cong"] \arrow[d, "\epsilon^i_*"] & K_0(C_{L,c}(\iota \otimes \pi)) \arrow[d, "\epsilon^i_*"] \\ KK(A, B) \arrow[r, "\cong"] & K_0(C_{L,c}(\pi)) \end{tikzcd}$$

for $i \in \{0, 1\}$. Homotopy invariance of KK-theory gives that the maps $\epsilon^0_*, \epsilon^1_* : KK(A, C[0,1] \otimes B) \to KK(A, B)$ are the same, whence the maps $\epsilon^0_*, \epsilon^1_* : K_0(C_{L,c}(1 \otimes \pi)) \to K_0(C_{L,c}(\pi))$ are too. On the other hand each ϵ^i induces maps $\epsilon^i : Q_L(1 \otimes \pi) \to Q_L(\pi)$ and $\epsilon^i : C_{L,c}(1 \otimes \pi; \mathcal{K}) \to C_{L,c}(\pi; \mathcal{K})$, and therefore induces a map $D_{M_C}(Q_L(1 \otimes \pi)) \to D_M(Q_L(1 \otimes \pi))$. All this gives rise to a commutative diagram

$$\begin{tikzcd}
C_{L,c}(1\otimes\pi) \arrow[r] \arrow[d, "\epsilon^i"] & Q_L(1\otimes\pi) \arrow[r] \arrow[d, "\epsilon^i"] & D_{M_C}(Q_L(1\otimes\pi)) \arrow[d, "\epsilon^i"] \\
C_{L,c}(\pi) \arrow[r] & Q_L(\pi) \arrow[r] & D_M(Q_L(\pi))
\end{tikzcd}$$

where the first pair of horizontal maps are the canonical quotients, and the second pair are the inclusions $a \mapsto (a,0)$. The horizontal maps induce isomorphisms on K-theory by Corollary 3.11 (first pair), and Lemmas 4.11 and 4.13 (second pair). Hence the maps $\epsilon^0_*, \epsilon^1_* : K_0(D_{M_C}(Q_L(1\otimes\pi))) \to K_0(D_M(Q_L(\pi)))$ are the same. We thus see that

$$[p^0, e] = \epsilon^0_*[p,e] = \epsilon^1_*[p,e] = [p^1, e],$$

which is the statement needed for well-definedness.

We now show that the map in the statement it is a homomorphism. Indeed, for $p, q \in \mathcal{P}^\pi(A,B)$, the element $[s_1ps_1^* + s_2qs_2^*]$ of $KK^\pi_{\mathcal{P}}(A,B)$ gets sent to

$$[s_1ps_1^* + s_2qs_2^*, e] = [s_1ps_1^* + s_2qs_2^*, s_1es_1^* + s_2es_2^*],$$

where we have used that $s_1s_1^* + s_2s_2^* = 1$ and that s_1, s_2 commute with e. As $s_1xs_1^*$ is orthogonal to $s_1ys_2^*$ for any x, y we have that

$$[s_1ps_1^* + s_2qs_2^*, s_1es_1^* + s_2es_2^*] = [s_1ps_1^*, s_1es_1^*] + [s_2qs_2^*, s_2es_2^*]$$

and as conjugation by s_1 and s_2 has no effect on K-theory by Lemma 3.8, this equals

$$[p,e] + [q,e],$$

which is the sum of the images of $[p]$ and $[q]$.

We now show that the map is surjective. Using Lemma 4.13, an arbitrary element of $K_0(D_M(Q_L(\pi)))$ can be represented as a class $[p,q]$ with p, q projections in $M_n(M)$ for some n, and with $p - q \in M_n(Q_L(\pi))$. We have that $[1-q, 1-q] = 0$ by Lemma 4.13, and thus $[p,q] = [p \oplus 1-q, q \oplus 1-q]$. The matrix $u = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}$ is a unitary in $M_{2n}(M)$, whence conjugating by (u,u) we see that

$$[p,q] = [p\oplus 1-q, q\oplus 1-q] = [u(p\oplus q)u^*, u(q\oplus 1-q)u^*] = [u(p\oplus q)u^*, 1_n\oplus 0_n],$$

where 1_n and 0_n are the unit and zero in $M_n(M)$. Choose now $2n$ isometries $v_1, ..., v_n, ..., v_{2n}$ in $\mathcal{B}(\mathbb{C}^2\otimes\ell^2)\subseteq\mathcal{L}$ such that $\sum_{i=1}^n v_i v_i^* = e$ and $\sum_{i=1}^{2n} v_i v_i^* = 1_{2n}$. The matrix

$$v := \begin{pmatrix} v_1 & v_2 & \cdots & v_{2n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_{2n}(M)$$

is an isometry, whence conjugation by (v,v) induces the trivial map on $K_0(D_M(Q_L(\pi)))$ by Lemma 3.8 and so

$$[p,q] = [vu(p\oplus q)u^*v^*, v(1_n\oplus 0)v^*] = [r\oplus 0_{2n-1}, e\oplus 0_{2n-1}],$$

where $r\in M$ is a projection such that $a := r-e$ is in $Q_L(\pi)$. We may lift a to a self-adjoint element $b\in C_{L,c}(\pi;\mathcal{K})$ by Lemma 3.13. Consider the self-adjoint element $(b+e,e)\in D_{M(C_{L,c}(\pi;\mathcal{K}))}(C_{L,c}(\pi;\mathcal{K}))$, which maps to $(r,e)\in D_M(Q_L(\pi))$ under the $*$-homomorphism

$$D_{M(C_{L,c}(\pi;\mathcal{K}))}(C_{L,c}(\pi;\mathcal{K})) \to D_M(Q_L(\pi))$$

induced by the quotient map $C_{L,c}(\pi;\mathcal{K})\to Q_L(\pi)$ of Lemma 3.13. Note that if $f:\mathbb{R}\to[-1,1]$ is the function defined by

$$f(t) := \begin{cases} 1 & t>1 \\ t & -1\leqslant t\leqslant 1 \\ -1 & t<-1 \end{cases}$$

then in $D_{M(C_{L,c}(\pi;\mathcal{K}))}(C_{L,c}(\pi;\mathcal{K}))$

$$f(b,e) = (f(b+e), f(e)) = (f(b+e), e),$$

and this element still maps to (r,e) by naturality of the functional calculus. Set $c = f(b+e)$. Then c is an element of $\mathcal{P}^\pi(A,B)$ such that $[c,e] = [r,e] = [p,q]$, so we are done with surjectivity.

To see injectivity, say $[p] \in KK^{\pi}_{\mathcal{P}}(A,B)$ is such that $[p,e]$ is zero in $K_0(D_M(Q_L(\pi)))$. In particular, $[p,e] = [e,e]$ by Lemma 4.13, and therefore there is a projection $(q_1,q_2) \in M_n(D_M(Q_L(\pi)))$ and a homotopy $p_{(1)} = (p^s_{(1)})_{s\in[0,1]}$ between $(p\oplus q_1, e\oplus q_2)$ and $(e\oplus q_1, e\oplus q_2)$ in $M_{n+1}(D_M(Q_L(\pi)))$. We will manipulate this homotopy to build a homotopy between p and e in $\mathcal{P}^{\pi}(A,B)$.

- Replacing $p_{(1)}$ by $p_{(2)} := p_{(1)} \oplus (q_2, q_1)$, we get a homotopy between $(p\oplus q_1\oplus q_2, e\oplus q_2\oplus q_1)$ and $(e\oplus q_1\oplus q_2, e\oplus q_2\oplus q_1)$.

- As $q_1 - q_2 \in M_n(Q_L(\pi))$, we get a homotopy
$$s \mapsto \left(p\oplus q_1\oplus q_2, e\oplus \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix} \begin{pmatrix} q_2 & 0 \\ 0 & q_1 \end{pmatrix} \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix}\right)$$
between $(p\oplus q_1\oplus q_2, e\oplus q_2\oplus q_1)$ and $(p\oplus q_1\oplus q_2, e\oplus q_1\oplus q_2)$, and similarly between $(e\oplus q_1\oplus q_2, e\oplus q_2\oplus q_1)$ and $(e\oplus q_1\oplus q_2, e\oplus q_1\oplus q_2)$. Concatenating these with the homotopy $p_{(2)}$ gives a homotopy $(p^s_{(3)})_{s\in[0,1]}$ between $(p\oplus q_1\oplus q_2, e\oplus q_1\oplus q_2)$ and $(e\oplus q_1\oplus q_2, e\oplus q_1\oplus q_2)$.

- Setting $r = q_1 \oplus q_2$ and replacing $p_{(3)}$ with
$$p^s_{(4)} := p_{(3)} \oplus \big((1-r),(1-r)\big)$$
gives a homotopy between $(p\oplus r\oplus (1-r), e\oplus r\oplus 1-r\oplus 0_{4n})$ and $(e\oplus r\oplus (1-r), e\oplus r\oplus (1-r))$.

- Set $u = \begin{pmatrix} r & 1-r \\ 1-r & r \end{pmatrix}$, which is a unitary in $M_{4n}(M)$. Moreover, u is self-adjoint, so connected to the identity via some path $(u^s)_{s\in[0,1]}$ of unitaries. Then
$$(1\oplus u^s, 1\oplus u^s)\big(p\oplus r\oplus (1-r), e\oplus r\oplus (1-r)\big)(1\oplus u^s, 1\oplus u^s)^*$$
defines a homotopy between $\big(p\oplus r\oplus (1-r), e\oplus r\oplus (1-r)\big)$ and $(p\oplus 1_{2n}\oplus 0_{2n}, e\oplus 1_{2n}\oplus 0_{2n})$. Similarly, we get a homotopy between $\big(e\oplus r\oplus (1-r), e\oplus r\oplus (1-r)\big)$ and $(e\oplus 1_{2n}\oplus 0_{2n}, e\oplus 1_{2n}\oplus 0_{2n})$. Concatenating these with $p_{(4)}$ gives a homotopy $p_{(5)}$ between $(p\oplus 1_{2n}\oplus 0_{2n}, e\oplus 1_{2n}\oplus 0_{2n})$ and $(e\oplus 1_{2n}\oplus 0_{2n}, e\oplus 1_{2n}\oplus 0_{2n})$.

- Write $p_{(5)}^s = (p_0^s, p_1^s)$ for paths of projections $(p_0^s)_{s\in[0,1]}$ and $(p_1^s)_{s\in[0,1]}$ in $M_{4n+1}(M)$. Then Lemma 4.8 gives a continuous path of unitaries $(v^s)_{s\in[0,1]}$ in $M_{4n+1}(M)$ with $v^0 = 1$, and $p_1^s = v_s(e \oplus 1_{2n} \oplus 0_{2n})v_s^*$ for all $s \in [0,1]$. Note in particular that $v_1(e \oplus 1_{2n} \oplus 0_{2n})v_1^* = (e \oplus 1_{2n} \oplus 0_{2n})$, even though we may not have that $v^1 = 1$. Define then

  $$p_{(6)}^s := (v_s, v_s)^* p_{(5)}^s (v_s, v_s),$$

  which gives a new homotopy between $(p \oplus 1_{2n} \oplus 0_{2n}, e \oplus 1_{2n} \oplus 0_{2n})$ and $(e \oplus 1_{2n} \oplus 0_{2n}, e \oplus 1_{2n} \oplus 0_{2n})$ with the additional property of being constant in the second variable.

- Let $M_{1\times 4n}(M)$ be the $1 \times 4n$ row matrices, and choose an isometry $w \in M_{1\times 4n}(\mathcal{B}(\ell^2)) \subseteq M$ be such that $w(1_{2n} \oplus 0_{2n})w^* = s_2 e s_2^*$. Define

  $$t := \begin{pmatrix} s_1 & w \end{pmatrix} \in M_{1\times 4n+1}(\mathcal{B}(\ell^2)) \subseteq M_{1\times 4n+1}(M)),$$

  which is an isometry, and define $p_{(7)}^s := t p_{(6)}^s t^*$. Then this is a homotopy in $D_M(Q_L(\pi))$ between $(s_1 p s_1^* + s_2 e s_2^*, s_1 e s_1^* + s_2 e s_2^*)$ and $(s_1 e s_1^* + s_2 e s_2^*, s_1 e s_1^* + s_2 e s_2^*)$ that is constant in the second variable.

Now, restricting the homotopy $p_{(7)}$ to the first variable gives a homotopy of projections in M, say $(p^s)_{s\in[0,1]}$ in M between $s_1 p s_1^* + s_2 s_2^* e$ and e, and such that $p^s - e$ is in $Q_L(\pi)$ for all s. The function

$$[0,1] \to D_M(Q_L(\pi)), \quad s \mapsto (p^s, e)$$

defines an idempotent, say q, in $C[0,1] \otimes D_M(Q_L(\pi))$. As the natural $*$-homomorphism

$$C[0,1] \otimes D_{M(C_{L,c}(\pi;\mathcal{K}))}(C_{L,c}(\pi;\mathcal{K})) \to C[0,1] \otimes D_M(Q_L(\pi))$$

is surjective, q lifts to a self-adjoint contraction of the form

$$(a, e) \in C[0,1] \times D_{M(C_{L,c}(\pi;\mathcal{K}))}(C_{L,c}(\pi;\mathcal{K}))$$

analogously to the argument at the end of the surjectivity half. The element a defines a homotopy in $\mathcal{P}^\pi(A, B)$ between $s_1ps_1^* + s_2es_2^*$ and e. On the other hand, $s_1ps_1^* + s_2es_2^* \sim p$ by Corollary 4.10, whence we have

$$p \sim s_1ps_1^* + s_2s_2^*e \sim e,$$

completing the proof that $[p] = 0$, and so we have injectivity.

To complete the proof, note that the existence of a canonical isomorphism $KK^\pi_{\mathcal{P}}(A, B) \cong KK(A, B)$ follows by combining: the isomorphism $KK^\pi_{\mathcal{P}}(A, B) \cong K_0(D_M(Q_L(\pi)))$ proved above; the isomorphism $K_*(D_M(Q_L(\pi))) \cong K_*(Q_L(\pi))$ of Lemma 4.13; the isomorphism $K_*(Q_L(\pi)) \cong K_*(C_{L,c}(\pi))$ of Corollary 3.11; and the isomorphism $K_0(C_{L,c}(\pi)) \cong KK(A, B)$ of Theorem 3.2. $\square$

Finally in this section we prove a technical lemma about functoriality that we will need later.

Lemma 4.15. *Let $\pi : A \to \mathcal{L}(E)$ be a substantially absorbing representation on a Hilbert B-module, and let $C = C_0(Y)$ be a separable and commutative C^*-algebra. For $y \in Y$, let $e^y : C_0(Y) \to \mathbb{C}$ be the $*$-homomorphism defined by evaluation at y. Let $\phi_B : KK(A, B) \to KK^\pi_{\mathcal{P}}(A, B)$ be the isomorphism of Proposition 4.14. Then if p is an element of $\mathcal{P}^{1\otimes\pi}(A, C \otimes B)$ with corresponding family $(p^y_t)_{t\in[1,\infty),y\in Y}$ as in Lemma 4.4, we have that*

$$e^y_*(\phi^{-1}_{C\otimes B}[p]) = \phi^{-1}_B[p^y].$$

Proof. The map

$$\mathcal{P}^{1\otimes\pi}(A, C \otimes B) \to \mathcal{P}^\pi(A, B), \quad p \mapsto p^y$$

induces a homomorphism

$$e^y_* : KK^{1\otimes\pi}_{\mathcal{P}}(A, C \otimes B) \to KK^\pi_{\mathcal{P}}(A, B).$$

Moreover, with notation as in the first paragraph of the proof of Theorem 4.14, e^y induces $*$-homomorphisms

$$e^y : Q_L(1 \otimes \pi) \to Q_L(\pi) \quad \text{and} \quad e^y : D_{M_C}(Q_L(1 \otimes \pi)) \to D_M(Q_L(\pi))$$

in the natural ways. Consider the diagram

$$\begin{tikzcd}
KK(A, C \otimes B) \arrow[r] \arrow[d, "e^y_*"] & K_0(C_{L,c}(1 \otimes \pi)) \arrow[r] \arrow[d, "e^y_*"] & K_0(Q_L(1 \otimes \pi)) \arrow[r] \arrow[d, "e^y_*"] & K_0(D_{M_C}(Q_L(1 \otimes \pi))) \arrow[r] \arrow[d, "e^y_*"] & KK^{1\otimes\pi}_{\mathcal{P}}(A, C \otimes B) \arrow[d, "e^y_*"] \\
KK(A, B) \arrow[r] & K_0(C_{L,c}(\pi)) \arrow[r] & K_0(Q_L(\pi)) \arrow[r] & K_0(D_M(Q_L(\pi))) \arrow[r] & KK^{\pi}_{\mathcal{P}}(A, B)
\end{tikzcd}$$

where: the first pairs of horizontal arrows are the isomorphisms of Theorem 3.2; the second pair of horizontal arrows are induced by the canonical quotient map; the third pair of horizontal arrows are induced by the inclusion $a \mapsto (a, 0)$; and the last pair of horizontal arrows are the isomorphisms of Theorem 4.14. The first square commutes by Lemma 3.17 (using also Proposition 2.9 to see that the representation $1 \otimes \pi$ is strongly absorbing). It is straightforward to see that the remaining squares commute: we leave this to the reader. As the isomorphisms $\phi_{C\otimes B}$ and ϕ_B are by definition the compositions of the arrows on the top and bottom rows respectively, the result follows. □

5 The topology on KK

Throughout this section, A and B are separable C^*-algebras.

Our goal in this section is to recall the canonical topology on $KK(A, B)$, and describe it in terms of the isomorphism $KK(A, B) \cong KK^{\pi}_{\mathcal{P}}(A, B)$ of Theorem 4.14.

We need a quantitative version of Definition 4.3; this will also be important to us later when we define our controlled KK-theory groups. See Definition 4.1 for graded representations and the neutral projection e used in the next definitions.

Definition 5.1. Let A and B be separable C^*-algebras, and let $\pi : A \to \mathcal{L}(E)$ be a graded representation on a Hilbert B-module. Let X be a finite subset of the unit ball A_1 of A, and let $\epsilon > 0$. Define $\mathcal{P}^{\pi}_{\epsilon}(X, B)$ to be the set of self-adjoint contractions in $\mathcal{L}(E)$ satisfying the following conditions:

(i) $p - e$ is in $\mathcal{K}(E)$;

(ii) $\|[p, a]\| < \epsilon$ for all $a \in X$;

(iii) $\|a(p^2 - p)\| < \epsilon$ for all $a \in X$.

For the next definition, see Definition 4.3 for the notation $\mathcal{P}^\pi(A, B)$.

Definition 5.2. Let A and B be separable C^*-algebras, and let $\pi : A \to \mathcal{L}(E)$ be a graded representation on a Hilbert B-module. For a finite subset X of A_1 and $\epsilon > 0$, define a function $\tau_{X,\epsilon} : \mathcal{P}^\pi(A, B) \to [1, \infty)$ by

$$\tau_{X,\epsilon}(p) := \inf\{t_0 \in [1, \infty) \mid p_t \in \mathcal{P}^\pi_\epsilon(X, B) \text{ for all } t \geqslant t_0\}.$$

For each $p \in \mathcal{P}^\pi(A, B)$, define $U(p; X, \epsilon)$ to be the subset of $\mathcal{P}^\pi(A, B)$ consisting of all q such that there exists $t \geqslant \max\{\tau_{X,\epsilon}(p), \tau_{X,\epsilon}(q)\}$ and a norm continuous path $(p^s)_{s\in[0,1]}$ in $\mathcal{L}(E)$ such that each p^s is in $\mathcal{P}^\pi_\epsilon(X, B)$, and with endpoints $p^0 = p_t$ and $p^1 = q_t$.

For the next definition, recall the homotopy equivalence relation $\sim$ on $\mathcal{P}^\pi(A, B)$ from Definition 4.5.

Lemma 5.3. *Let $\pi : A \to \mathcal{L}(E)$ be a graded representation of A on a graded Hilbert B-module. Let $p \in \mathcal{P}^\pi(A, B)$, X be a finite subset of A_1, and $\epsilon > 0$. Then:*

(i) if $p' \sim p$, then $U(p; X, \epsilon) = U(p'; X, \epsilon)$;

(ii) if $q \in U(p; X, \epsilon)$ and $q \sim q'$, then $q' \in U(p; X, \epsilon)$.

Proof. Part (ii) follows from part (i) on noting that q is in $U(p; X, \epsilon)$ if and only if p is in $U(q; X, \epsilon)$. It thus suffices to prove (i).

Assume then that $p \sim p'$, so there is a homotopy $(p^s)_{s\in[0,1]}$ in $\mathcal{P}^{1\otimes\pi}(A, C[0, 1]\otimes B)$ be a homotopy between p and p'. The definition of a homotopy gives $t_p \geqslant \max\{\tau_{\epsilon,X}(p), \tau_{\epsilon,X}(p')\}$ such that $p^s_{t_p}$ is in $\mathcal{P}^\pi_\epsilon(X, B)$ for all $s \in [0, 1]$. Let q be an element of $U(p; X, \epsilon)$, and let $t_q \geqslant \{\tau_{X,\epsilon}(q), \tau_{X,\epsilon}(p)\}$ be such that there is a homotopy $(q^s)_{s\in[0,1]}$ connecting p_{t_q} and q_{t_q}. Write I for whichever of the intervals $[t_p, t_q]$ or $[t_q, t_p]$ makes sense. Then concatenating the homotopies $(p^s_{t_p})_{s\in[0,1]}$, $(p_t)_{t\in I}$ and $(q^s)_{s\in[0,1]}$ shows that q is in $U(p'; X, \epsilon)$. Hence $U(p; X, \epsilon) \subseteq U(p'; X, \epsilon)$. The opposite inclusion follows by symmetry. □

Definition 5.4. Let $\pi : A \to \mathcal{L}(E)$ be a graded representation on a Hilbert B-module. For a finite subset X of A_1, $\epsilon > 0$, and $[p] \in KK^{\pi}_{\mathcal{P}}(A, B)$, define the X-ϵ *neighbourhood* of p to be

$$V([p]; X, \epsilon) := \{[q] \in KK_{\mathcal{P}}(A, B) \mid q \in U(p; X, \epsilon)\}.$$

(note that $V([p]; X, \epsilon)$ does not depend on the particular representative of the class $[p]$ by Lemma 5.3). The *asymptotic topology* on $KK^{\pi}_{\mathcal{P}}(A, B)$ is the topology generated by the sets $V([p]; X, \epsilon)$ as X ranges over finite subsets of A_1, ϵ over $(0, \infty)$, and p over $\mathcal{P}^{\pi}(A, B)$.

Lemma 5.5. *Let $\pi : A \to \mathcal{L}(E)$ be a graded representation on a Hilbert B-module. For any $[p] \in KK^{\pi}_{\mathcal{P}}(A, B)$, the collection of sets $V([p]; X, \epsilon)$ as X ranges over finite subsets of A_1 and ϵ over $(0, \infty)$ form a neighbourhood base of $[p]$. Moreover, the asymptotic topology is first countable.*

Proof. Using Lemma 5.3, it suffices to prove the corresponding statements for the topology on $\mathcal{P}^{\pi}(A, B)$ generated by the sets $U(p; X, \epsilon)$, so we do this instead.

For the neighbourhood base claim, we must show that whenever $q_1, ..., q_n$, $X_1, ..., X_n$ and $\epsilon_1, ..., \epsilon_n$ are such that $p \in \bigcap_{i=1}^{n} U(q_i; X_i, \epsilon_i)$, then there exist X, ϵ with

$$U(p; X, \epsilon) \subseteq \bigcap_{i=1}^{n} U(q_i; X_i, \epsilon_i).$$

As whenever $Y \supseteq X$ and $\delta \leqslant \epsilon$, we have that $U(p; Y, \delta) \subseteq U(p; X, \epsilon)$, it suffices to prove this for $n = 1$. Assume then we are given $q \in \mathcal{P}^{\pi}(A, B)$, a finite subset $X \subseteq A_1$, and $\epsilon > 0$ such that $p \in U(q; X, \epsilon)$. We claim that $U(p; X, \epsilon) \subseteq U(q; X, \epsilon)$, which will suffice to complete the neighbourhood base part of the proof. Indeed, say r is in $U(p; X, \epsilon)$. Then there exists $t_r \geqslant \max\{\tau_{X,\epsilon}(p), \tau_{X,\epsilon}(r)\}$ and a homotopy $(r^s)_{s\in[0,1]}$ passing through $\mathcal{P}^{\pi}_{\epsilon}(X, B)$ connecting p_{t_r} and r_{t_r}. Similarly, there exists $t_q \geqslant \max\{\tau_{X,\epsilon}(p), \tau_{X,\epsilon}(q)\}$ and a homotopy $(q^s)_{s\in[0,1]}$ passing through $\mathcal{P}^{\pi}_{\epsilon}(X, B)$ connecting q_{t_q} and p_{t_q}. Let I be the closed interval bounded by t_r and t_q. Then concatenating the three paths $(q^s)_{s\in[0,1]}$, $(p_t)_{t\in I}$, and $(r^s)_{s\in[0,1]}$ shows that r is in $U(q; X, \epsilon)$, so we are done.

Assume now that A is separable, so in particular, there exists a nested sequence $X_1 \subseteq X_2 \subseteq$ of finite subsets of the unit ball A_1 with dense union. Fix a point $p \in \mathcal{P}^\pi(A, B)$. We claim that the sets $U(p; X_n, 1/n)$ form a neighbourhood basis at p. Indeed, given what we have already proved, it suffices to show that for any finite $X \subseteq A_1$ and any $\epsilon > 0$ there exists n with $U(p; X_n, 1/n) \subseteq U(p; X, \epsilon)$. Let n be so large so that for all $a \in X$ there is $a' \in X_n$ with $\|a - a'\| < \epsilon/2$, and also so that $1/n < \epsilon/2$. From the choice of n, it follows that $\mathcal{P}^\pi_{1/n}(X_n, B) \subseteq \mathcal{P}^\pi_\epsilon(X, B)$, from which the inclusion $U(p; X_n, 1/n) \subseteq U(p; X, \epsilon)$ follows. $\square$

We now recall the canonical topology on $KK(A, B)$, which has been introduced and studied in different pictures by several authors: see for example the discussion in [5] for some background and references. Dadarlat[18] showed in [5, Lemma 3.1] that this topology is characterized by the following property (and used this to show that the various different descriptions that had previously appeared in the literature agree).

Lemma 5.6. *Let A and B be separable C^*-algebras. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one point compactification of the natural numbers, and for each $n \in \overline{\mathbb{N}}$, let $e^n : C(\overline{\mathbb{N}}, B) \to B$ be the $*$-homomorphism defined by evaluation at n. Then the canonical topology on $KK(A, B)$ is characterized by the following conditions.*

(i) It is first countable.

(ii) A sequence (x_n) in $KK(A, B)$ converges to x_∞ in $KK(A, B)$ if and only if there is an element $x \in KK(A, C(\overline{\mathbb{N}}, B))$ such that $e^n_*(x) = x_n$ for all $n \in \overline{\mathbb{N}}$. $\square$

Theorem 5.7. *Let $\pi : A \to \mathcal{L}(E)$ be a substantially absorbing representation. Then the isomorphism of Theorem 4.14 is a homeomorphism between the asymptotic topology on $KK^\pi_{\mathcal{P}}(A, B)$ and the canonical topology on $KK(A, B)$.*

We need an ancillary lemma.

[18] Dadarlat attributes some of the idea here to unpublished work of Pimsner.

Lemma 5.8. *Let $\pi : A \to \mathcal{L}(E)$ be a graded representation on a Hilbert B-module. For any $\epsilon > 0$ and any finite $X \subseteq A_1$, if $p, q \in \mathcal{P}^{\pi}_{\epsilon/2}(X, B)$ satisfy $\|p - q\| < \epsilon/6$, then there exists a homotopy $(p^s)_{s\in[0,1]}$ connecting p and q and passing through $\mathcal{P}^{\pi}_{\epsilon}(X, B)$.*

Proof. A straight line homotopy between p and q works. We leave the details to the reader. □

Proof of Theorem 5.7. We have already see that the asymptotic topology is first countable in Lemma 5.5. Hence by Lemma 5.6, it suffices to show that sequential convergence in the asymptotic topology is characterized by condition (ii) from Lemma 5.6.

Say first that $([p^n])_{n\in\overline{\mathbb{N}}}$ is a collection of elements of $KK^{\pi}_{\mathcal{P}}(A, B)$. Let $1\otimes\pi$ be the amplification of π to the Hilbert $C(\overline{\mathbb{N}}) \otimes B$ module $C(\overline{\mathbb{N}}) \otimes E$, and let $q \in \mathcal{P}^{1\otimes\pi}(A, C(\overline{\mathbb{N}}, B))$ be such that for all $n \in \overline{\mathbb{N}}$ we have $e^n_*[q] = [p^n]$. We want to show that the sequence $([p^n])_{n\in\mathbb{N}}$ converges to $[p^\infty]$ in the asymptotic topology. For this, it follows from Lemmas 5.3 and 5.5 that it suffices to fix a finite subset X of A_1 and $\epsilon > 0$, and show that p^n is in $U(p^\infty; X, \epsilon)$ for all suitably large n.

Recall the function $\tau_{X,\epsilon}$ of Definition 5.2. As q is an element of $\mathcal{P}^{1\otimes\pi}(A, C(\overline{\mathbb{N}}, B))$, the number $\tau := \sup_{n\in\overline{\mathbb{N}}} \tau_{X,\epsilon/2}(q^n)$ is finite. As q is in $\mathcal{P}^{1\otimes\pi}(A, C(\overline{\mathbb{N}}, B))$ we also see from Lemma 4.4 that there exists N such that $\|q^n_\tau - q^\infty_\tau\| < \epsilon/6$ for all $n \geqslant N$. We claim that p^n is in $U(p^\infty; X, \epsilon)$ for all $n \geqslant N$, which will complete the first half of the proof.

Using Lemma 4.4, we may identify q with a collection $(q^n)_{n\in\overline{\mathbb{N}}}$ of elements of $\mathcal{P}(A, B)$ (satisfying certain conditions). Now let $n \geqslant N$ and consider the following homotopies.

(i) As $e^\infty_*[q] = [p^\infty]$, Theorem 4.14 implies that $q^\infty \sim p^\infty$, and thus there is $t_\infty \geqslant \max\{\tau, \tau_{X,\epsilon}(p^\infty)\}$ and a homotopy passing through $\mathcal{P}^{\pi}_{\epsilon}(X, B)$ and connecting $p^\infty_{t_\infty}$ and $q^\infty_{t_\infty}$.

(ii) Similarly to (i), there is $t_n \geqslant \max\{\tau, \tau_{X,\epsilon}(p^n)\}$ and a homotopy passing through $\mathcal{P}^{\pi}_{\epsilon}(X, B)$ and connecting $p^n_{t_n}$ and $q^n_{t_n}$.

(iii) As $\|q_\tau^n - q_\tau^\infty\| < \epsilon/6$ for all $n \geqslant N$ and as $\tau = \sup_{n\in\overline{\mathbb{N}}} \tau_{X,\epsilon/2}(q^n)$, Lemma 5.8 gives a homotopy passing through $\mathcal{P}_\epsilon^\pi(X, B)$ and connecting q_τ^∞ and q_τ^n.

(iv) The path $(q_t^n)_{t\in[\tau,t_n]}$ is a homotopy passing through $\mathcal{P}_\epsilon^\pi(X, B)$ that connects q_τ^n and $q_{t_n}^n$.

(v) The path $(q_t^\infty)_{t\in[\tau,t_\infty]}$ is a homotopy passing through $\mathcal{P}_\epsilon^\pi(X, B)$ that connects q_τ^∞ and $q_{t_\infty}^\infty$.

Now let $t_{\max} = \max\{t_n, t_\infty\}$. Concatenating the five homotopies above with the homotopies $(p_t^n)_{t\in[t_n,t_{\max}]}$ and $(p_t^\infty)_{t\in[t_\infty,t_{\max}]}$ (which pass through $P_\epsilon^\pi(X, B)$) shows that p^n is in $U(p^\infty; X, \epsilon)$ as claimed.

For the converse, fix a sequence $X_1 \subseteq X_2 \subseteq \cdots$ of nested finite subsets of A_1 with dense union. Let us assume that $([p^n])_{n\in\mathbb{N}}$ is a sequence in $KK_{\mathcal{P}}^\pi(A, B)$ that converges to $[p^\infty]$ in the asymptotic topology. We want to construct an element $q \in \mathcal{P}^{1\otimes\pi}(A, C(\overline{\mathbb{N}}, B))$ such that $e_*^n[q] = [p^n]$ for each n in $\overline{\mathbb{N}}$. We will define new representatives of the classes $[p^n]$ as follows. For each m, let $N_m \in \mathbb{N}$ be the smallest natural number such that p^n is in $U(p^\infty; X_m, 1/m)$ for all $n \geqslant N_m$; as $[p^n]$ converges to $[p^\infty]$ in the asymptotic topology, such an N_m exists, and the sequence $N_1, N_2, ...$ is non-decreasing.

Choose a sequence $t_1 \leqslant t_2 \leqslant \cdots$ in $[1, \infty)$ that tends to infinity in the following way. For $n < N_1$, let $t_n = 1$. Otherwise, let m be as large as possible so that $n \geqslant N_m$. Let $t_n = \max\{\tau_{X_m,1/m}(p^n), \tau_{X_m,1/m}(p^\infty), t_1, ..., t_{n-1}\} + 1$, and note the choice of N_m implies that $p^n \in U(p^\infty; X_m, 1/m)$, so there exists a homotopy between $p_{t_n}^n$ and $p_{t_n}^\infty$ parametrized as usual by $[0, 1]$ that passes through $\mathcal{P}_{1/m}^\pi(X_m, B)$. Approximating this homotopy by a piecewise-linear homotopy, we may assume that it is Lipschitz, and still passing through $\mathcal{P}_{1/m}^\pi(X_m, B)$. Moreover lengthening the interval parametrizing the homotopy, we may assume that it is 1-Lipshcitz. In conclusion, for some suitably large $r_n \in [1, \infty)$, we may assume that we have a 1-Lipschitz homotopy

$(p^{n,t})_{t\in[t_n,t_n+r_n]}$ between $p^n_{t_n}$ and $p^\infty_{t_n}$. Define for each $n \in \mathbb{N}$

$$q^n := \begin{cases} p^\infty_t & t \in [1, t_n] \\ p^{n,t} & t \in [t_n, t_n + r_n] \\ p^n_t & t \geqslant t_n + r_n \end{cases},$$

and note that $[q^n] = [p^n]$ for all $n \in \mathbb{N}$ using Lemma 4.6. Define $q^\infty = p^\infty$. Using the characterization of Lemma 4.4, one checks directly that $q = (q^n)_{n\in\overline{\mathbb{N}}}$ defines an element of $\mathcal{P}^{1\otimes\pi}(A, C(\overline{\mathbb{N}}, B))$. This element satisfies $e^n_*[q] = [p^n]$ by construction, so we are done. $\square$

6 Controlled KK-theory and KL-theory

As usual, A and B always denote separable C^*-algebras.

Our goal in this section is to (finally!) introduce our controlled KK-theory groups, and describe $KL(A, B)$ in terms of an their inverse limit. For the next definition, recall the notion of a graded representation from Definition 4.1, and the set $\mathcal{P}^\pi_\epsilon(X, B)$ from Definition 5.1.

Definition 6.1. Let $\pi : A \to \mathcal{L}(E)$ be a graded representation. Let $X \subseteq A_1$ be finite and let $\epsilon > 0$. Equip $\mathcal{P}_\epsilon(X, B)$ with the norm topology it inherits from $\mathcal{L}(E)$, and define $KK^\pi_\epsilon(X, B) := \pi_0(\mathcal{P}^\pi_\epsilon(X, B))$ to be the associated set of path components.

In the special case that π is substantial we show below that the set $KK^\pi_\epsilon(X, B)$ has a natural abelian group structure. In this case, we call the groups $KK^\pi_\epsilon(X, B)$ *controlled KK-theory groups*.

Our first goal is to define a group structure on $KK^\pi_\epsilon(X, B)$. For this, let us assume that (π, E) is substantial (see Definition 4.1), and fix two isometries $s_1, s_2 \in \mathcal{B}(\ell^2)$ satisfying the Cuntz relation $s_1s_1^* + s_2s_2^* = 1$. Using the inclusion $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ from Lemma 4.2, we think of s_1 and s_2 as isometries in $\mathcal{L}(E)$ that commute with the subalgebra A and the neutral projection e. We define an operation on $KK_\epsilon(X, B)$ by

$$[p] + [q] := [s_1ps_1^* + s_2qs_2^*].$$

The following lemma is proved in exactly the same way as Lemma 4.7.

Lemma 6.2. *With notation as above, the set $KK_\epsilon^\pi(X,B)$ is a commutative semigroup. The group operation does not depend on the choice of s_1 and s_2.* $\square$

In order to prove that $KK_\epsilon^\pi(X,B)$ is a monoid, we need an analogue of Lemma 4.9.

Lemma 6.3. *Let $\pi : A \to \mathcal{L}(E)$ be a substantial representation on a Hilbert B-module. Let X be a finite subset of A_1, let $\epsilon > 0$, let p be an element of $\mathcal{P}_\epsilon^\pi(X,B)$, and let v be an isometry in the canonical copy of $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ from Lemma 4.2. Then the formula*

$$vpv^* + (1 - vv^*)e$$

defines an element of $\mathcal{P}_\epsilon^\pi(X,B)$ in the same path component as p.

Proof. The fact that $vpv^* + (1 - vv^*)e$ is an element of $\mathcal{P}_\epsilon^\pi(X,B)$ follows from the fact that v commutes with A. We fix $\delta \in (0,1)$, to be determined in the course of the proof by X, p, and ϵ. As $p - e$ is in $\mathcal{K}(E)$, there exists an infinite rank projection $r \in \mathcal{B}(\ell^2)$ such that $1 - r$ also has infinite rank, and such that

$$\|(1-r)(p-e)\| < \delta. \tag{13}$$

Note that as r commutes with e, line (13) implies that

$$\|[r,p]\| < 2\delta. \tag{14}$$

As r is a projection and commutes with elements of A, and as p is a contraction, this implies that for any $a \in X$,

$$\|a((rpr)^2 - rpr)\| \leqslant \|r[p,r]pr\| + \|ra(p^2-p)r\| < 2\delta + \max_{a\in X}\|a(p^2-p)\|. \tag{15}$$

Define now $q := rpr + (1-r)e$, which is a self-adjoint contraction. Note that $q - e = rpr - re = r(p-e)r$, so $q - e$ is in $\mathcal{K}(E)$. We have $q^2 - q = (rpr)^2 - rpr$, and so line (15) implies that for all $a \in X$,

$$\|a(q^2-q)\| < 2\delta + \max_{a\in X}\|a(p^2-p)\|.$$

Moreover,

$$\|q - p\| = \|rpr - rp + (1-r)e - (1-r)p\| \leqslant \|[r,p]\| + \|(1-r)(p-e)\| < 3\delta$$

by lines (10) and (14). Hence as long as δ is so suitably small (depending on ϵ and $\epsilon - \max_{a\in X} \|a(p^2 - p)\|$), we see that q is in $\mathcal{P}^{\pi}_{\epsilon}(X, B)$. Moreover, for suitably small δ, we have that the path

$$[0,1] \mapsto \mathcal{L}(E), \quad s \mapsto sp - (1-s)q$$

is norm continuous and passes through $\mathcal{P}^{\pi}_{\epsilon}(X, B)$, and so shows that $p \sim q$. Hence it suffices to prove the result with p replaced by q.

Now, let $v \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ be an isometry as in the original statement. Choose a partial isometry $w \in \mathcal{B}(\ell^2)$ such that $ww^* = 1 - r$ and $w^*w = 1 - vv^* + v(1-r)v^*$; such exists as the operators appearing on the right hand sides of these equations are infinite rank projections. Define

$$u := vr + w^* \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E).$$

Then one checks that u is unitary, and moreover that $uqu^* = vqv^* + (1 - vv^*)$. Let $(u_s)_{s\in[0,1]}$ be any norm continuous path of unitaries in $\mathcal{B}(\ell^2)$ connecting u to the identity. Then if we write "$r \sim s$" to mean that $r, s \in \mathcal{P}^{\pi}_{\epsilon}(X, B)$ are in the same path component, the homotopy $(u_s q u_s^*)_{s\in[0,1]}$ shows that $q \sim vqv^* + (1 - vv^*)$. In conclusion, we have that

$$p \sim q \sim vqv^* + (1 - vv^*)e \sim vpv^* + (1 - vv^*)e,$$

where the last '$\sim$' follows from the homotopy $\big(v(sp + (1-s)q)v^* + (1 - vv^*)e\big)_{s\in[0,1]}$. $\square$

Corollary 6.4. *Let $\pi : A \to \mathcal{L}(E)$ be a substantial representation on a Hilbert B-module. Let X be a finite subset of A_1, let $\epsilon > 0$. Then the commutative semigroup $KK^{\pi}_{\epsilon}(X, B)$ is a commutative monoid with identity element $[e]$.*

Proof. This follows from Lemma 6.2, and Lemma 6.3 with $v = s_1$ (whence $1 - vv^* = s_2 s_2^*$). $\square$

Proposition 6.5. *Let $\pi : A \to \mathcal{L}(E)$ be a substantial representation on a Hilbert B-module. Let X be a finite subset of A_1, let $\epsilon > 0$. Then the monoid $KK_\epsilon(X, B)$ is a group.*

Proof. We must show that inverses exist. Let then p be an element of $\mathcal{P}^\pi_\epsilon(X, B)$. Let $M_2(\mathbb{C})$ be unitally included in $\mathcal{L}$ as in Lemma 4.2, and let u be the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$, so $ueu = 1-e$. The self-adjoint contraction $s_1es_1^* + s_2u(1-p)us_2^*$ then defines an element of $\mathcal{P}^\pi_\epsilon(X, B)$, which we claim represents the inverse of $[p]$ in $KK^\pi_\epsilon(X, B)$. To establish the claim, we must show that

$$s_1(s_1es_1^* + s_2u(1-p)us_2^*)s_1^* + s_2ps_2^*$$

is homotopic to e through elements of $\mathcal{P}^\pi_\epsilon(X, B)$.

We first define

$$v := s_2s_1^*s_1^* + s_1s_1s_2^* + s_1s_2s_2^*s_1^*,$$

which is unitary in $\mathcal{B}(\ell^2) \subseteq \mathcal{L}$. Note that

$$v(s_1(s_1ps_1^*+s_2u(1-p)us_2^*)s_1^*+s_2ps_2^*)v^* = s_1(s_1ps_1^*+s_2u(1-p)us_2^*)s_1^*+s_2es_2^*.$$

Moreover, v is connected to the identity in the unitary group of $\mathcal{B}(\ell^2)$; as $\mathcal{B}(\ell^2)$ commutes with A and e, this shows that

$$s_1(s_1es_1^*+s_2u(1-p)us_2^*)s_1^*+s_2ps_2^* \quad \text{and} \quad s_1(s_1ps_1^*+s_2u(1-p)us_2^*)s_1^*+s_2es_2^*$$

are homotopic through elements of $\mathcal{P}^\pi_\epsilon(X, B)$. Lemma 6.3 (with $v = s_1$) implies that the second element $s_1(s_1ps_1^* + s_2u(1-p)us_2^*)s_1^* + s_2es_2^*$ above is homotopic in $\mathcal{P}^\pi_\epsilon(X, B)$ to $s_1ps_1^* + s_2u(1-p)us_2^*$, so it suffices to show that $s_1ps_1^* + s_2u(1-p)us_2^*$ is homotopic to e through elements of $\mathcal{P}^\pi_\epsilon(X, B)$.

Now, connect u to the identity through unitary elements of $M_2(\mathbb{C})$. This gives a path, say $(p_t^{(0)})_{t\in[0,1]}$ connecting $s_1ps_1^*+s_2u(1-p)us_2^*$ to $s_1ps_1^*+s_2(1-p)s_2^*$. We have $\|[p_t^{(0)}, a]\| < \epsilon$ and $\|a((p_t^{(0)})^2 - p_t^{(0)})\| < \epsilon$ for all t and all $a \in X$. At this point, to simplify notation, let us write elements of $\mathcal{L}(E)$ as 2×2

matrices with respect to the matrix units $e_{ij} := s_i s_j^*$. With this notation[19], consider the path $(p_t^{(1)})_{t\in[0,\pi/2]}$ defined by

$$p_t^{(1)} := \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix} \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

One computes that

$$(p_t^{(1)})^2 - p_t^{(1)} = \begin{pmatrix} p^2 - p & 0 \\ 0 & \cos^2(t)(p^2 - p) \end{pmatrix},$$

whence $\|a((p_t^{(1)})^2 - p_t^{(1)})\| < \epsilon$ for all $t \in [0,\pi/2]$ and all $a \in X$. Another computation gives that for any $a \in A$ and $t \in [0,\pi/2]$,

$$[a, p_t^{(1)}] = [a,p] \begin{pmatrix} \cos^2(t) & -\cos(t)\sin(t) \\ -\cos(t)\sin(t) & -\cos^2(t) \end{pmatrix}.$$

The norm of the matrix appearing on the right hand side is $|\cos(t)|$ (or just $\cos(t)$ for $t \in [0,\pi/2]$), and therefore we see that $\|[a, p_t^{(1)}\| < \epsilon$ for all $a \in X$ and all $t \in [0,\pi/2]$.

Concatenating the paths $(p_t^{(0)})_{t\in[0,1]}$ and $(p_t^{(1)})_{t\in[0,\pi/2]}$, and reparametrizing, we get a new path $(p_t)_{t\in[0,1]}$ connecting $s_1ps_1^* + s_2u(1-p)us_2^*$ and $s_1s_1^*$. This path does not define an element of $\mathcal{P}_\epsilon^\pi(X,B)$: it satisfies all the conditions to be in this set except that $p_t - e$ need not be in $\mathcal{K}(E)$. It remains to adjust the path $(p_t)_{t\in[0,1]}$ to get a path connecting $s_1ps_1^* + s_2u(1-p)us_2^*$ and e in $\mathcal{P}_\epsilon^\pi(X,B)$.

Let $\varpi : \mathcal{L}(E) \to \mathcal{L}(E)/\mathcal{K}(E)$ be the quotient map. With respect to the decomposition in Lemma 4.2, ϖ is injective on the canonical copy of $\mathcal{B}(\mathbb{C}^2 \otimes \ell^2) \subseteq \mathcal{L}(E)$. As $p - e \in \mathcal{K}(E)$ and e is a projection, we see that the path $(\varpi(p_t))_{t\in[0,1]}$ passes through projections in $\mathcal{B}(\mathbb{C}^2 \otimes \ell^2)$, and it connects e and $s_1s_1^*$. Hence using Lemma 4.8, there exists a continuous path of unitaries $(w_t)_{t\in[0,1]}$ in $\mathcal{B}(\mathbb{C}^2 \otimes \ell^2)$ with $w_0 = 1$ and such that $\varpi(p_t) = w_t\varpi(p_0)w_t^*$ for all t. The path $(w_t^*p_tw_t)_{t\in[0,1]}$ then lies in $\mathcal{P}_\epsilon^\pi(X,B)$, and connects $s_1ps_1^* + s_2u(1-p)us_2^*$ and e as required. $\square$

[19] In more formal notation, $p_t^{(1)} = s_1(p + \sin^2(t)(1-p))s_1^* + s_1\cos(t)\sin(t)(1-p)s_2^* + s_2\cos(t)\sin(t)(1-p)s_1^* + s_2\cos^2(t)(1-p)s_2^*$.

Having established that each $KK_\epsilon^\pi(X,B)$ is a group, we now arrange these groups into an inverse system, and show that the resulting inverse limit agrees with $KL(A,B)$.

Definition 6.6. Let $\mathcal{X}$ be the set of all pairs (X,ϵ) where X is a finite subset of A_1, and $\epsilon \in (0,\infty)$. We equip $\mathcal{X}$ with the partial order defined by $(X,\epsilon) \leqslant (Y,\delta)$ if for any graded representation $\pi : A \to \mathcal{L}(E)$ we have that $\mathcal{P}_\delta^\pi(Y,B) \subseteq \mathcal{P}^\epsilon(X,B)$.

Remark 6.7. We record some basic properties of the partially ordered set $\mathcal{X}$.

(i) Note that $(X,\epsilon) \leqslant (Y,\delta)$ if $X \subseteq Y$ and $\delta \leqslant \epsilon$.

(ii) It follows from this that $\mathcal{X}$ is directed: an upper bound for (X_1,ϵ_1) and (X_2,ϵ_2) is given by $(X_1 \cup X_2, \min\{\epsilon_1,\epsilon_2\})$.

(iii) The partial order in Definition 6.6 contains a lot more comparable pairs that those arising from the "naive ordering" on the set $\mathcal{X}$ defined by "$X \subseteq Y$ and $\delta \leqslant \epsilon$" as in (i) above. For example, the naive ordering never contains cofinal sequences (even for $A = \mathbb{C}$), while the ordering from Definition 6.6 always does. To see this, let $(a_n)_{n=1}^\infty$ be a dense sequence in A_1, and define $X_n := \{a_1, ..., a_n\}$. Then the sequence $(X_n, 1/n)_{n=1}^\infty$ is cofinal in $\mathcal{X}$ for the ordering from Definition 6.6.

(iv) If A is generated by some finite set $X \subseteq A_1$, then the sequence $(X,1/n)_{n=1}^\infty$ is also cofinal in $\mathcal{X}$.

(v) If $(X,\epsilon) \leqslant (Y,\delta)$ for the ordering from Definition 6.6, then there is a canonical "forget control" map

$$\varphi_{X,\epsilon}^{Y,\delta} : KK_\delta^\pi(Y,B) \to KK_\epsilon^\pi(X,B). \tag{16}$$

In this way, the collection $(KK_\epsilon^\pi(X,B))_{(X,\epsilon)\in\mathcal{X}}$ becomes an inverse system, with well-defined inverse limit $\varprojlim KK_\epsilon^\pi(X,B)$. Recall that the inverse limit can be concretely realized as the abelian group

$$\left\{ (x_{X,\epsilon}) \in \prod_{(X,\epsilon)\in\mathcal{X}} KK_\epsilon^\pi(X,B) \mid \varphi_{X,\epsilon}^{Y,\delta}(x_{Y,\delta}) = x_{X,\epsilon} \right\}. \tag{17}$$

It is equipped with a natural family of homomorphisms

$$\varpi_{X,\epsilon} : \lim_{\leftarrow} KK^{\pi}_{\epsilon}(X, B) \to KK^{\pi}_{\epsilon}(X, B)$$

defined by restricting to each coordinate.

(vi) Recall that the inverse limit has the following universal property. For any abelian group A equipped with a family of homomorphisms $\psi_{X,\epsilon} : A \to KK^{\pi}_{\epsilon}(X, B)$ such that the diagrams

$$\begin{tikzcd} & A \arrow[dl, "\psi_{Y,\delta}"'] \arrow[dr, "\psi_{X,\epsilon}"] & \\ KK^{\pi}_{\delta}(Y, B) \arrow[rr, "\varphi^{Y,\delta}_{X,\epsilon}"] & & KK^{\pi}_{\epsilon}(X, B) \end{tikzcd}$$

commute, there is a unique homomorphism $\varpi : A \to \lim_{\leftarrow} KK^{\pi}_{\epsilon}(X, B)$ making the following diagrams

$$\begin{tikzcd} & A \arrow[ddl, bend right, "\psi_{Y,\delta}"'] \arrow[ddr, bend left, "\psi_{X,\epsilon}"] & \\ & \lim_{\leftarrow} KK^{\pi}_{\epsilon}(X, B) \arrow[dl, "\varpi_{Y,\delta}"'] \arrow[dr, "\varpi_{X,\epsilon}"] & \\ KK^{\pi}_{\delta}(Y, B) \arrow[rr, "\varphi^{Y,\delta}_{X,\epsilon}"] & & KK^{\pi}_{\epsilon}(X, B) \end{tikzcd}$$

all commute.

(vii) Recall (this is straightforward to check from either the concrete definition above, or the universal property) that any cofinal subset of a directed subset of a directed set defines the same inverse limit. Hence we may compute $\lim_{\leftarrow} KK^{\pi}_{\epsilon}(X, B)$ using the cofinal sequences from parts (iii) or (iv).

Our goal in the remainder of this section is to show that

$$\lim_{\leftarrow} KK_{\epsilon}(X, B) \cong KL(A, B)$$

whenever π is substantially absorbing as in Definition 4.1.

For the next lemma, recall the notation $\tau_{X,\epsilon}(p)$ from Definition 5.2 above and the notation $KK^{\pi}_{\mathcal{P}}(A, B)$ from Definition 4.5.

Lemma 6.8. *Let $\pi : A \to \mathcal{L}(E)$ be a substantial representation on a Hilbert B-module. For each $(X, \epsilon) \in \mathcal{X}$ there is a group homomorphism*

$$\pi_{X,\epsilon} : KK^{\pi}_{\mathcal{P}}(A, B) \to KK^{\pi}_{\epsilon}(X, B)$$

defined by sending $[p]$ to the class of $[p_t]$, where $t = \tau_{X,\epsilon}(p) + 1$. Moreover, the family of homomorphisms $(\pi_{X,\epsilon})_{(X,\epsilon)\in\mathcal{X}}$ are compatible with the forgetful maps in line (16) above in the sense that the diagrams

$$\begin{tikzcd} KK^{\pi}_{\mathcal{P}}(A,B) \arrow[r, equal] \arrow[d, "\pi_{Y,\delta}"] & KK^{\pi}_{\mathcal{P}}(A,B) \arrow[d, "\pi_{X,\epsilon}"] \\ KK^{\pi}_{\delta}(Y,B) \arrow[r, "\varphi^{Y,\delta}_{X,\epsilon}"] & KK^{\pi}_{\epsilon}(X,B) \end{tikzcd}$$

commute, and thus determine a group homomorphism

$$\varpi : KK^{\pi}_{\mathcal{P}}(A, B) \to \varprojlim KK_{\epsilon}(X, B).$$

Proof. To see that the map $\pi_{X,\epsilon}$ is well-defined, let $(p^s)_{s\in[0,1]}$ be a homotopy between $p^0, p^1 \in \mathcal{P}^{\pi}(A, B)$. Let $t_0 = \tau_{X,\epsilon}(p^0) + 1$, $t_1 = \tau_{X,\epsilon}(p^1) + 1$, and choose t_2 such that $t_2 \geqslant \max\{t_0, t_1\}$, and such that $p^s_{t_2}$ is in $\mathcal{P}^{\pi}_{\epsilon}(X, B)$ for all s. Then concatenating the homotopies $(p^0_t)_{t\in[t_0,t_2]}$, $(p^s_{t_2})_{s\in[0,1]}$, and $(p^1_t)_{t\in[t_1,t_2]}$ shows that $p^0_{t_0} \sim p^1_{t_1}$ in $\mathcal{P}^{\pi}_{\epsilon}(X, B)$ and we get well-definedness.

To see that $\pi_{X,\epsilon}$ is a group homomorphism, let $s_1, s_2 \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ be a pair of isometries satisfying the Cuntz relation, and used to define the group operations on both $KK^{\pi}_{\mathcal{P}}(A, B)$ and $KK^{\pi}_{\epsilon}(X, B)$, and let $[p], [q] \in \mathcal{P}^{\pi}(A, B)$. Then $[p] + [q]$ is the class of $[s_1ps_1^* + s_2qs_2^*]$, and we have that

$$\pi_{X,\epsilon} : [s_1ps_1^* + s_2qs_2^*] \mapsto [s_1p_{t_{p+q}}s_1^* + s_1q_{t_{p+q}}s_1^*]$$

where $t_{p+q} := \tau_{X,\epsilon}(s_1ps_1^* + s_2qs_2^*) + 1$. On the other hand, if we define $t_p := \tau_{X,\epsilon}(p) + 1$ and $t_q := \tau_{X,\epsilon}(q)$, then

$$\pi_{X,\epsilon}[p] + \pi_{X,\epsilon}[q] = [s_1p_{t_p}s_1^* + s_2q_{t_q}s_2^*].$$

Define $t_{p+q} := \max\{t_p, t_q\}$, and say without loss of generality that $t_p \geqslant t_q$. Then the path $(s_1p_{t_p}s_1^* + s_2q_ts_2^*)_{t\in[t_q,t_p]}$ shows that $\pi_{X,\epsilon}([p] + [q]) = \pi_{X,\epsilon}[p] + \pi_{X,\epsilon}[q]$ as required.

Compatibility of the maps $\pi_{X,\epsilon}$ with the forgetful maps in line (16) is proved via similar arguments; we leave the details to the reader. The existence of ϖ follows from this and the universal property of the inverse limit as in Remark 6.7, part (vi). $\square$

Using the ideas in the previous section, we now get the promised relationship to KL. To state it, let $\overline{\{0\}}$ be the closure of zero in the asymptotic topology on $KK^{\pi}_{\mathcal{P}}(A,B)$ from Definition 5.4. Following Dadarlat[20] [5, Section 5], define $KL(A,B)$ to be the quotient of $KK(A,B)$ by the closure of the singleton $\{0\}$ in the canonical topology.

Theorem 6.9. *Let $\pi : A \to \mathcal{L}(E)$ be a substantial representation on a Hilbert B-module. The homomorphism ϖ in Lemma 6.8 is surjective and descends to a well-defined isomorphism*

$$\varpi : \frac{KK^{\pi}_{\mathcal{P}}(A,B)}{\overline{\{0\}}} \cong \lim_{\leftarrow} KK^{\pi}_{\epsilon}(X,B).$$

In particular, if π is substantially absorbing, there is a canonical isomorphism $\lim\limits_{\leftarrow} KK^{\pi}_{\epsilon}(X,B) \cong KL(A,B)$.

Proof. It follows directly from the definitions that an element of $KK^{\pi}_{\mathcal{P}}(A,B)$ is in the closure of $\{0\}$ if and only if it maps to zero under ϖ, so it remains to show that ϖ is surjective. For this, let us choose a cofinal sequence $(X_n, 1/n)_{n=1}^{\infty}$ of $\mathcal{X}$ as in Remark 6.7 part (iii), whence there is a canonical isomorphism

$$\lim_{\leftarrow} KK^{\pi}_{\epsilon}(X,B) = \lim_{\leftarrow} KK^{\pi}_{1/n}(X_n,B)$$

as in Remark 6.7 part (vii). Hence it suffices to prove surjectivity of the induced map

$$KK^{\pi}_{\mathcal{P}}(A,B) \to \lim_{\leftarrow} KK^{\pi}_{1/n}(X_n,B).$$

Using the concrete definition of the inverse limit from line (17) above, let $([p^n])_{n=1}^{\infty}$ be a sequence defining an element of $\lim\limits_{\leftarrow} KK^{\pi}_{1/n}(X_n,B)$ with $p^n \in$

[20]The original definition of KL is due to Rørdam [18, page 434]: Rørdam's definition applies when A satisfies the UCT, and the two definitions agree in that case.

$\mathcal{P}^{\pi}_{1/n}(X_n, B)$ for each n. As this sequence defines an element of the inverse limit we must have that for each n, the forgetful map

$$KK^{\pi}_{1/(n+1)}(X_{n+1}, B) \to KK^{\pi}_{1/n}(X_n, B)$$

sends $[p^{n+1}]$ to $[p^n]$. This implies that there is a homotopy $(p^n_s)_{s\in[0,1]}$ of elements in $\mathcal{P}^{\pi}_{1/n}(X_n, B)$ with $p^n_0 = p^n$ and $p^n_1 = p^{n+1}$. Define $p : [1, \infty) \to \mathcal{L}$ by setting $p_t := p^n_{t-n}$ whenever t is in $[n, n+1]$, and note that p is then an element of $\mathcal{P}^{\pi}(A, B)$.

We claim that $\pi[p] = ([p^n])_{n=1}^{\infty}$, which will complete the proof. Indeed, it suffices to fix n and show that $\pi_{X_n,1/n}[p] = [p^n]$. We have $\pi_{X_n,1/n}[p] = [p_{t_p}]$, where $t_p := \tau_{X_n,1/n}(p)$. By definition of p and of $\tau_{X_n,1/n}$, the interval I with endpoints n and t_p is such that the path $(p_t)_{t\in I}$ lies entirely in $\mathcal{P}^{\pi}_{1/n}(X_n, B)$. Hence

$$\pi_{X_n,1/n}[p] = [p_{t_p}] = [p_n] = [p^n]$$

and we are done with the first isomorphism in the statement.

The isomorphism $\varprojlim KK^{\pi}_{\epsilon}(X, B) \cong KL(A, B)$ is a direct consequence of this, Theorem 4.14, and Theorem 5.7. $\square$

7 Identifying the closure of zero

As usual, A and B are separable C^*-algebras throughout this section.

Our goal in this section is to concretely identify the closure of zero in the asymptotic topology on $KK^{\pi}_{\mathcal{P}}(A, B)$. This will complete the proof of Theorem 1.1 from the introduction.

We will need an analogue of Lemma 4.4 in the controlled setting.

Lemma 7.1. *Let $\pi : A \to \mathcal{L}(E)$ be a graded representation on a Hilbert B-module. Let $C = C_0(Y)$ be a separable commutative C^*-algebra, and let $C \otimes E$ be equipped with the amplified representation $1 \otimes \pi$ of A as in the discussion just above Lemma 4.4. Let $\epsilon > 0$, and let X of the unit ball A_1 of A. Then there is a natural identification between elements p of $\mathcal{P}^{1\otimes\pi}_{\epsilon}(X, C \otimes B)$ and parametrized families of self-adjoint contractions $(p^y)_{y\in Y}$ such that the corresponding function $p : Y \to \mathcal{L}(E)$ has the following properties:*

(i) the function $p - e$ is in $C_0(Y, \mathcal{K}(E))$;

(ii) $\|[p, a]\|_{C_b(Y,\mathcal{L}(E))} < \epsilon$ for all $a \in X$;

(iii) $\|a(p^2 - p)\|_{C_b(Y,\mathcal{L}(E))} < \epsilon$ for all $a \in X$.

Proof. Analogously to Lemma 4.4, the proof rests on the identification $\mathcal{K}(C \otimes E) = C_0(Y, \mathcal{K}(E))$; we leave the details to the reader. $\square$

Let now $SB = C_0(0, 1) \otimes B$ denote the suspension of B. The following lemma is immediate from Lemma 7.1.

Corollary 7.2. *Let $\pi : A \to \mathcal{L}(E)$ be a graded representation of A. For any finite subset X of A_1 and $\epsilon > 0$, elements of $\mathcal{P}_\epsilon^{1\otimes\pi}(X, SB)$ can be identified with norm continuous functions*

$$p : [0, 1] \to \mathcal{L}(E), \quad t \mapsto p_t$$

such that:

(i) $p_0 = p_1 = e$;

(ii) $p_t - e \in \mathcal{K}(E)$ for all $t \in [0, 1]$;

(iii) $\|a(p_t^2 - p_t)\| < \epsilon$ and $\|[p_t, a]\| < \epsilon$ for all $a \in X$ and all $t \in [0, 1]$. $\square$

Definition 7.3. Let $\pi : A \to \mathcal{L}(E)$ be a graded representation of A, and let $X \subseteq A_1$ be finite and $\epsilon > 0$. Let $p, q \in \mathcal{P}_\epsilon^\pi(X, SB)$ be represented by paths as in Corollary 7.2, and define their *concatenation* $p \cdot q$ to be the path that follows p then q: precisely

$$(p \cdot q)_t := \begin{cases} p_{2t} & 0 \leqslant t \leqslant 1/2 \\ q_{2t-1} & 1/2 < t \leqslant 1 \end{cases} .$$

The following lemma says that the group operation on $KK_\epsilon(X, SB)$ can equivalently be defined by concatenation; it will be useful later in the section.

Lemma 7.4. *Let $\pi : A \to \mathcal{L}(E)$ be a substantial representation of A, let X be a finite subset of A_1, $\epsilon > 0$, and let SB be the suspension of B. Then for any $[p], [q] \in KK^{\pi}_{\epsilon}(X, SB)$ we have that $[p] + [q] = [p \cdot q]$. Moreover, $-[p]$ is represented by the path $\overline{p}$ that traverses p in the opposite direction.*

Proof. Up to homotopy, we may assume that $p_t = e$ for all $t \in [1/3, 1]$, and that $q_t = e$ for all $t \in [0, 2/3]$. The sum $[p] + [q]$ is then represented by a function of the form

$$(s_1 p s_1^* + s_2 q s_2^*)_t = \begin{cases} s_1 p_t s_1^* + s_2 e s_2^* & t \in [0, 1/3] \\ e & t \in [1/3, 2/3] \\ s_1 e s_1^* + s_2 q_t s_2^* & t \in [2/3, 1] \end{cases} .$$

Let $v = s_1 s_2^* + s_2 s_1^*$, which is a unitary in $\mathcal{B}(\ell^2)$. As the unitary group of $\mathcal{B}(\ell^2)$ is connected, there is a path $u = (u_t)_{t \in [0,1]}$ of unitaries in $\mathcal{B}(\ell^2)$ such that $u_t = 1$ for $t \leqslant 1/3$ and $u_t = v$ for $t \geqslant 2/3$. We may consider u as an element of the unitary group of $\mathcal{L}(C_0(0,1) \otimes \ell^2) \subseteq \mathcal{L}(C_0(0,1) \otimes E)$; using that u commutes with e, we have then that

$$(u(s_1 p s_1^* + s_2 q s_2^*) u^*)_t = \begin{cases} s_1 p_t s_1^* + s_2 e s_2^* & t \in [0, 1/3] \\ e & t \in [1/3, 2/3] \\ s_1 q_t s_1^* + s_2 e s_2^* & t \in [2/3, 1] \end{cases} . \tag{18}$$

On the other hand, the unitary group of $\mathcal{L}(C_0(0,1) \otimes \ell^2)$ is connected (even contractible) by [4, Theorem on page 433], and commutes with both e and A, so we may connect u to the identity via some norm continuous path in this unitary group. This gives a homotopy showing that $u(s_1 p s_1^* + s_2 q s_2^*) u^*$ defines the same element of $KK^{\pi}_{\epsilon}(X, SB)$ as $s_1 p s_1^* + s_2 q s_2^*$. From the description in line (18), we have also that $u(s_1 p s_1^* + s_2 q s_2^*) u^*$ and $s_1 (p \cdot q) s_1^* + s_2 e s_2^*$ define the same element of $KK^{\pi}_{\epsilon}(X, SB)$. This element is homotopic to $p \cdot q$ by Lemma 6.3, so we are done with the proof that $[p] + [q] = [p \cdot q]$.

The fact that $-[p] = [\overline{p}]$ is a consequence of the first part: indeed, $[p] + [\overline{p}] = [p \cdot \overline{p}]$, and $p \cdot \overline{p}$ is easily seen to be homotopic to the constant path e, which represents the identity in $KK^{\pi}_{\epsilon}(X, SB)$ by Corollary 6.4. ∎

We now recall the definition of the $\varprojlim^1$ group of the inverse system of Remark 6.7, part (v). For simplicity[21], we choose a cofinal sequence $(X_n, \epsilon_n)_{n=1}^{\infty}$ of the directed set $\mathcal{X}$ of Definition 6.6: for example, the cofinal sequences of Remark **??**, parts (iii) or (iv) will work. Note that for any such sequence, we must have that $\epsilon_n \to 0$ as $n \to \infty$.

Remark 7.5. It will turn out (as a consequence of Theorem 7.7 below) that the $\varprojlim^1$ group does not depend on the choice of cofinal system. As a result, we will sometimes be a little sloppy and write something like "$\varprojlim^1 KK_\epsilon(X, B)$" without specifying a choice of cofinal sequence.

Let then $\pi : A \to \mathcal{L}(E)$ be a substantial representation, and consider the inverse system associated to our cofinal sequence

$$\cdots \stackrel{\varphi_{n+1}}{\to} KK^{\pi}_{\epsilon_n}(X_n, B) \stackrel{\varphi_n}{\to} KK^{\pi}_{\epsilon_{n-1}}(X_{n-1}, B) \stackrel{\varphi_{n-1}}{\to} \cdots \stackrel{\varphi_1}{\to} KK^{\pi}_{\epsilon_1}(X_1, B)$$

where the maps φ_n are the forget control maps of Remark 6.7 part (v). Consider the homomorphism

$$\alpha : \prod_{n=1}^{\infty} KK^{\pi}_{\epsilon_n}(X_n, B) \to \prod_{n=1}^{\infty} KK^{\pi}_{\epsilon_n}(X_n, B), \quad (x_n) \mapsto (\varphi_n(x_n)).$$

As in Remark 6.7 part (v), the inverse limit is concretely realized as the kernel of the homomorphism $\mathrm{id} - \alpha$. On the other hand, the group $\varprojlim^1 KK^{\pi}_{\epsilon_n}(X_n, B)$ is by definition the *cokernel* of $\mathrm{id} - \alpha$.

Lemma 7.6. *Let $\pi : A \to \mathcal{L}(E)$ be a substantial representation. Consider an element $([p^n])_{n=1}^{\infty}$ of the product $\prod_n KK^{\pi}_{\epsilon_n}(X_n, SB)$. Use the identification in Lemma 7.1 to consider each p^n as a function $p^n : [0,1] \to \mathcal{L}(E)$, and define $p : [1, \infty) \to \mathcal{L}(E)$ by setting $p_t := p^n_{t-n}$ whenever $t \in [n, n+1]$. Then p is in $\mathcal{P}^{\pi}(A, B)$, and the formula*

$$\psi : \prod_n KK^{\pi}_{\epsilon_n}(X_n, SB) \to KK^{\pi}_{\mathcal{P}}(A, B), \quad ([p^n])_{n=1}^{\infty} \mapsto [p]$$

[21] This is not strictly necessary, but we could not find a convenient treatment of $\varprojlim^1$ groups associated to inverse limit functors over arbitrary directed sets in the literature, and it takes a little work to exhume the facts we need from any treatment we could find.

gives a well-defined homomorphism. Moreover, this homomorphism takes image in the closure $\overline{\{0\}}$ of the zero element for the asymptotic topology (Definition 5.4), and descends to a well-defined homomorphism

$$\psi : \lim_{\leftarrow}{}^1 KK^{\pi}_{\epsilon_n}(X_n, SB) \to KK^{\pi}_{\mathcal{P}}(A, B)$$

on the $\lim^1$*-group.*

Proof. We leave the direct check that p is an element of $\mathcal{P}^{\pi}(A, B)$ to the reader (for this purpose, recall that $\epsilon_n \to 0$ as $n \to \infty$). To see that ψ is well-defined on the product $\prod_n KK^{\pi}_{\epsilon_n}(X_n, SB)$, let $([p^n])_{n=1}^{\infty}$ and $([q^n])_{n=0}^{\infty}$ be sequences in $KK^{\pi}_{\epsilon_n}(X_n, SB)$ representing the same class in the product $\prod_n KK^{\pi}_{\epsilon_n}(X_n, SB)$, and with images $[p]$ and $[q]$ in $KK^{\pi}_{\mathcal{P}}(A, B)$. Then for each n there is a homotopy $(p^{n,s})_{s\in[0,1]}$ between them. Using the identification in Lemma 4.4, define a new function $p : [1, \infty) \to \mathcal{L}(C[0, 1] \otimes E)$ by $p^s_t := p^{n,s}_{t-n}$ for $t \in [n, n+1]$. Then direct checks using the conditions in Lemma 4.4 show that $(p^s)_{s\in[0,1]}$ is a homotopy between p and q, whence $[p] = [q]$ and we have well-definedness. The fact that ψ is a homomorphism follows directly, as we may assume the group operations are all defined using the same pair of isometries (s_1, s_2) satisfying the Cuntz relation.

To see now that the image of the map is contained in $\overline{\{0\}}$, we must show that if $[p]$ is in the image, then for every finite subset $X \subseteq A_1$ and $\epsilon > 0$ there is $t \geqslant \tau_{X,\epsilon}(p)$ (see Definition 5.2) and a homotopy passing through $\mathcal{P}^{\pi}_{\epsilon}(X, B)$ connecting p_t to e. This is clear, however: by construction of p, there is a sequence (t_n) tending to infinity such that $p_{t_n} = e$ for all n, and we can use cofinality of our sequence (X_n, ϵ_n) in the directed set $\mathcal{X}$ of Definition 6.6 to construct the required homotopies.

For the statement about the $\lim^1$ group, we must show that if $([p^n])_{n=1}^{\infty}$ is a sequence in $\prod_n KK^{\pi}_{\epsilon_n}(X_n, SB)$, then the image of $([p^n])$ is the same as that of the sequence $([p^{n+1}])_{n=1}^{\infty}$. Indeed, say the image of the former is p and the image of the latter is q. Then by construction we have that $q_t = p_{t+L}$ for all t and some fixed L. The path $(p^s)_{s\in[0,1]}$ defined by $p^s_t := p_{t+sL}$ is a homotopy between p and q, so we are done. □

We are now ready for the main result of this section. As already commented above, it completes the proof of Theorem 1.1.

Theorem 7.7. *Let $\pi : A \to \mathcal{L}(E)$ be a substantial representation. Then the map*

$$\psi : \lim_{\leftarrow}{}^1 KK^{\pi}_{\epsilon_n}(X_n, SB) \to KK^{\pi}_{\mathcal{P}}(A, B)$$

from Lemma 7.6 is an isomorphism onto the closure of zero in $KK^{\pi}_{\mathcal{P}}(A, B)$.

Proof. To see that the map is onto, let $p \in \mathcal{P}^{\pi}(A, B)$ be an element so that $[p]$ is in the closure of zero. Using the description of neighbourhood bases from Lemma 5.5, we may find an increasing sequence (t_n) in $[1, \infty)$ such that $t_n \to \infty$, such that $t_n \geqslant \tau_{X_n,\epsilon_n}(p)$ for all n, and such that for each n there is a homotopy $(q^n_s)_{s\in[0,1]}$ such that $q^n_0 = p_{t_n}$ and $q^n_1 = e$, and such that q^n_s is in $\mathcal{P}^{\pi}_{\epsilon_n}(X_n, B)$ for all s. For each n, build a path $r^n : [0, 1] \to \mathcal{L}(E)$ by concatenating the paths $(q^n_{1-s})_{s\in[0,1]}$, $(p_t)_{t\in[t_n,t_{n+1}]}$, and $(q^{n+1}_s)_{s\in[0,1]}$, and reparametrizing to get the domain equal to $[0, 1]$. Note that the path $(r^n_s)_{s\in[0,1]}$ starts and ends at e, and has image contained in $\mathcal{P}^{\pi}_{\epsilon_n}(X_n, B)$. One checks directly that r^n lies in $\mathcal{P}^{\pi}_{\epsilon_n}(X_n, SB)$ using the conditions in Corollary 7.2, and thus we get a class $([r^n]) \in \lim_{\leftarrow}{}^1 KK^{\pi}_{\epsilon_n}(X_n, SB)$. We claim the image of $([r^n])$ in $KK^{\pi}_{\mathcal{P}}(A, B)$ is $[p]$.

Indeed, up to reparametrizations (which do not affect the resulting class in $KK^{\pi}_{\mathcal{P}}(A, B)$), the image of $([r^n])$ is represented by concatenating the paths

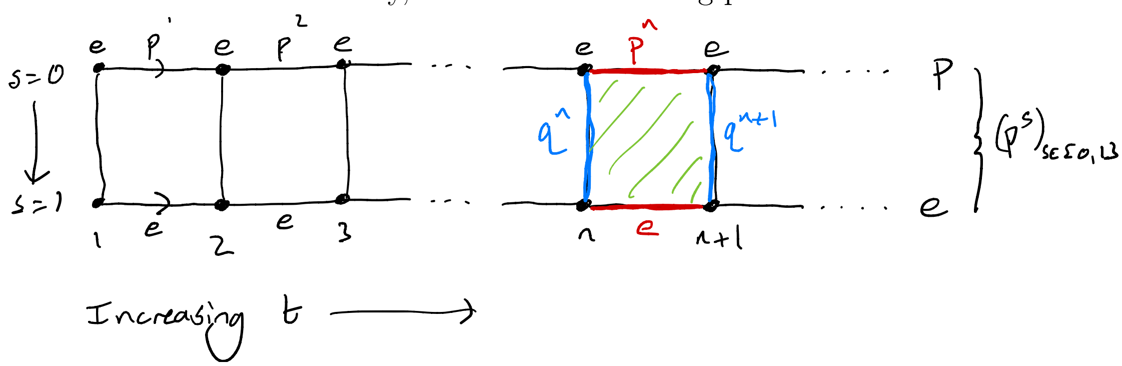
$$(q^1_{1-s})_{s\in[0,1]},\ (p_t)_{t\in[t_1,t_2]},\ (q^2_s)_{s\in[0,1]},\ (q^2_{1-s})_{s\in[0,1]},\ (p_t)_{t\in[t_2,t_3]},$$
$$(q^3_s)_{s\in[0,1]}, (q^3_{1-s})_{s\in[0,1]},\ (p_t)_{t\in[t_3,t_4]},\ \dots$$

As each pair $(q^n_s)_{s\in[0,1]}$, $(q^n_{1-s})_{s\in[0,1]}$ consists of the same path traversed in opposite directions, a homotopy removes all these pairs, so we are left with the concatenation of the paths

$$(q^1_{1-s})_{s\in[0,1]},\ (p_t)_{t\in[t_1,t_2]},\ (p_t)_{t\in[t_2,t_3]},\ (p_t)_{t\in[t_3,t_4]},\ \dots$$

or in other words of $(q^1_{1-s})_{s\in[0,1]}$ and $(p_t)_{t\geqslant t_1}$. As any element $q \in \mathcal{P}^{\pi}(A, B)$ is homotopic to the path defined by $t \mapsto q_{t+L}$ for any fixed $L > 0$, this path is homotopic to the original p and we are done with surjectivity.

For injectivity, let $([p^n])_{n=1}^{\infty}$ be a sequence in $\prod_n KK^{\pi}_{\epsilon_n}(X_n, SB)$ that maps to zero in $KK_{\mathcal{P}}(A,B)$, so there is a homotopy $(p^s)_{s\in[0,1]}$ connecting the resulting image p to e. Here p is the result of concatenating the functions $p^n : [0,1] \to \mathcal{L}$, so for $t \in [n, n+1]$, $p_t = p^n_{t-n}$. For each n, let $(q^n)_{s\in[0,1]}$ be the path defined by $q^n_s := p^s_n$, which defines an element of $\mathcal{P}^{\pi}_{\epsilon}(X, SB)$ for some ϵ and X. Schematically, we have the following picture:



For each n, let $m(n)$ be the largest natural number such that the elements $(p^s_t)_{s\in[0,1],t\in[n,n+1]}$ are all in $\mathcal{P}^{\pi}_{\epsilon_{m(n)}}(X_{m(n)}, B)$. Note that $m(n) \to \infty$ as $n \to \infty$ by definition of a homotopy, ad that q^n_s is in $\mathcal{P}^{\pi}_{\epsilon_{m(n)}}(X_{m(n)}, B)$ for all n.

Now, for each n, consider the element $-[q^n] + [p^n] + [q^{n+1}]$, which is in $KK^{\pi}_{\epsilon_{m(n)}}(X_{m(n)}, SB)$ by choice of $m(n)$. This element is represented by the concatenation $\overline{q^n} \cdot p^n \cdot q^{n+1}$ by Lemma 7.4, so it forms three sides of the 'square' $(p^s_t)_{s\in[0,1],t\in[n,n+1]}$ (pictured as the green square in the diagram above). The fourth side is the constant function with value e, so $-[q^n]+[p^n]+[q^{n+1}] = [e]$ in $KK^{\pi}_{\epsilon_{m(n)}}(X_{m(n)}, SB)$. Moreover, $[e] = 0$ by Corollary 6.4, so

$$[p^n] = [q^n] - [q^{n+1}] \quad \text{for all} \quad KK_{1/m(n)}(X_{m(n)}, SB) \tag{19}$$

for all n.

We claim that the existence of the elements q^n satisfying the formulas in line (19) shows that original element $([p^n])_{n=1}^{\infty}$ is zero in the $\lim^1$-group, which will complete the proof. Let $\alpha : \prod_k KK_{\epsilon_k}(X_k, SB) \to \prod_k KK^{\pi}_{\epsilon_k}(X_k, SB)$ be the map defined by shifting down one unit, so the $\lim^1$ group is by definition the cokernel of $1-\alpha$. Choose a subsequence $(n_l)_{l=1}^{\infty}$ of the natural numbers such that the sequence $(m(n_l))_{l=1}^{\infty}$ is strictly increasing, which exists as $m(n) \to \infty$ as $n \to \infty$.

For each $l \in \mathbb{N}$, define an element $x^n \in \prod_k KK^{\pi}_{\epsilon_k}(X_k, SB)$ by setting the component x^n_k in $KK^{\pi}_{1/k}(X_k, SB)$ to be

$$x^n_k := \begin{cases} [p^n] & n_l \leqslant n < n_{l+1},\ m(n_l) < k \leqslant n \\ 0 & \text{otherwise} \end{cases}.$$

Note then that

$$(1-\alpha)(x^n) = (0, ..., 0, \underbrace{[p^n]}_{n^{\text{th}}}, 0, ...) - (0, ..., 0, \underbrace{[p^n]}_{m(n_l)^{\text{th}}}, 0, ...).$$

Define

$$x := \sum_{n=0}^{\infty} x^n \in \prod_k KK^{\pi}_{\epsilon_k}(X_k, SB);$$

this makes sense as the sum is finite in each component $KK^{\pi}_{\epsilon_k}(X_k, SB)$ using that $m(n) \to \infty$ as $n \to \infty$. We have then that $(1-\alpha)(x)$ is the element whose k^{th} component is the difference of the elements $([p^n])_{n=n_1}^{\infty}$ and y, where the k^{th} component of y is

$$\sum_{\{n \mid n_l \leqslant n < n_{l+1},\, m(n_l)=k\}} [p^n]$$

(with the empty sum being interpreted as zero). Noting that the difference between $([p^n])_{n=n_1}^{\infty}$ and $([p^n])_{n=1}^{\infty}$ is in the image of $1-\alpha$ (as indeed is any element with only finitely many non-zero terms), it thus suffices to prove that y is in the image of $1-\alpha$.

For this, for each $l \in \mathbb{N}$, define $z^l \in \prod_k KK^{\pi}_{\epsilon_k}(X_k, SB)$ to be the element with k^{th} component z^n_k defined by

$$z^l_k := \begin{cases} [q^{m(n_{l+1})}] & m(n_l) < k \leqslant m(n_{l+1}) \\ 0 & \text{otherwise} \end{cases}.$$

We have then that $(1-\alpha)(z^l)$ has entries: $-[q^{m(n_{l+1})}]$ in the $m(n_l)^{\text{th}}$ place; $[q^{m(n_{l+1})}]$ in the $m(n_{l+1})^{\text{th}}$ place; and zero elsewhere. As before, we have $z := \sum_{l=0}^{\infty} z^l$ makes sense. It follows from the above discussion that $(1-\alpha)(z)$ has entries given by

$$((1-\alpha)z)_k = \begin{cases} [q^{m(n_l)}] - [q^{m(n_{l+1})}] & k = m(n_l) \text{ for some } l \\ 0 & \text{otherwise} \end{cases}.$$

On the other hand, we have that

$$[q^{m(n_l)}] - [q^{m(n_{l+1})}] = \sum_{n=n_l}^{n_{l+1}-1} [q^n] - [q^{n+1}] = \sum_{n=n_l}^{n_{l+1}-1} [p^n].$$

Hence $(1-\alpha)(z) = y$, and we are done. $\square$

A Alternative cycles for controlled KK-theory

In this appendix, we discuss some technical variants of the groups $KK_\epsilon^\pi(X, B)$ that will be useful in the sequel to this work. Throughout this section, A and B are separable C^*-algebras. We will typically assume that A is unital.

As usual, throughout the appendix A and B are separable C^*-algebras, and a representation $\pi : A \to \mathcal{L}(E)$ is assumed to take values in the adjointable operators on a Hilbert B-module E.

A.1 Controlled KK-groups in the unital case

In this subsection, we specialize to the unital case and give a picture of the controlled KK-groups in this case. The basic point is that in this case one can use honest projections to define these groups rather than just elements satisfying $\|a(p^2 - p)\| < \epsilon$ for suitable $a \in A$ and $\epsilon > 0$.

Let A be a unital C^*-algebra, and let $\pi : A \to \mathcal{L}(E)$ be a representation of A. We write π_1 for the corestriction of the representation to a representation on $\pi(1_A) \cdot E$. Note that if π is substantial (see Definition 4.1), then π_1 is too.

Definition A.1. Let A and B be separable C^*-algebras, and let $\pi : A \to \mathcal{L}(E)$ be a graded representation of A with associated neutral projection e as in Definition 4.1. Let X be a finite subset of the unit ball A_1 of A, and let $\epsilon > 0$. Define $\mathcal{P}_\epsilon^{\pi,p}(X, B)$ to be the set of projections in $\mathcal{L}(E)$ satisfying the following conditions:

(i) $p - e$ is in $\mathcal{K}(E)$

(ii) $\|[p, a]\| < \epsilon$ for all $a \in X$.

Equip $\mathcal{P}^{\pi,p}_\epsilon(X,B)$ with the norm topology it inherits from $\mathcal{L}(E)$, and define $KK^{\pi,p}_\epsilon(X,B) := \pi_0(\mathcal{P}^{\pi,p}_\epsilon(X,B))$, i.e. $KK^{\pi,p}_\epsilon(X,B)$ is the set of path components of $\mathcal{P}^{\pi,p}_\epsilon(X,B)$.

Definition A.2. Let $\pi : A \to \mathcal{L}(E)$ be a substantial representation of A, and let $KK^{\pi,p}_\epsilon(X,B)$ be as in Definition A.1. Let $s_1, s_2 \in \mathcal{B}(\ell^2)$ be a pair of isometries satisfying the Cuntz relation $s_1s_1^* + s_2s_2^* = 1$, considered as elements of $\mathcal{L}(E)$ via the inclusion $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ of Lemma 4.2.

Define a binary operation on $KK^{\pi,p}_\epsilon(X,B)$ by

$$[p] + [q] := [s_1ps_1^* + s_2qs_2^*]$$

(it is clear that this definition respects path components, so really does define an operation on $KK^{\pi,p}_\epsilon(X,B)$).

To show that $KK^{\pi,p}_\epsilon(X,B)$ is a group, we will need an analog of Lemma 6.3. Write "$\sim$" for the equivalence relation

Lemma A.3. *Fix notation as in Definition A.2. Let $e \in \mathcal{L}(E)$ be the neutral projection, let p be an element of $\mathcal{P}^{\pi,p}_\epsilon(X,B)$, and let v be an isometry in the canonical copy of $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ from Lemma 4.2. Then the formula*

$$vpv^* + (1 - vv^*)e$$

defines an element of $\mathcal{P}^{\pi,p}_\epsilon(X,B)$ that is in the same path component as p.

Proof. The proof is essentially the same as that of Lemma 6.3, so we just give a brief sketch, pointing out differences where necessary. As in the proof of Lemma 6.3, we fix $\delta > 0$, and choose an infinite rank projection $r \in \mathcal{B}(\ell^2)$ such that $\|(1-r)(p-e)\| < \delta$ just as in that proof. Let $\chi : \mathbb{R} \to \{0,1\}$ be the characteristic function of $(1/2, \infty)$, and define $q := \chi(rpr + (1-r)e)$, which is an element of $\mathcal{P}^{\pi,p}_\epsilon(X,B)$ by the computations in the proof of Lemma 6.3, at least for suitably small δ. Moreover, for δ suitably small, the homotopy

$$[0,1] \to \mathcal{L}(E), \quad s \mapsto \chi(sp + (1-s)q) \tag{20}$$

shows that p and q define the same class in $\mathcal{P}^{\pi,p}_\epsilon(X,B)$ (here we use that there is some $\gamma = \gamma(\delta)$ such that $\gamma \to 0$ as $\delta \to 0$, and such that $\|\chi(sp +$

$(1-s)q) - p\| < \gamma$ for all s). The proof is now finished analogously to that of Lemma 6.3 by considering the element $u := vr + w^* \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ defined just as in that proof, using the element q above in place of the element q from the proof of Lemma 6.3, and using the homotopy in line (20) where the homotopy $s \mapsto sp + (1-q)$ is used in the proof of Lemma 6.3. $\square$

Lemma A.4. *Fix notation as in Definition A.2. Then $KK_\epsilon^{\pi,p}(X,B)$ is an abelian group, and does not depend on the choice of Cuntz isometries s_1 and s_2.*

Proof. The fact that $KK_\epsilon^{\pi,p}(X,B)$ is an abelian semigroup with operation not depending on the choice of s_1, s_2 proceeds in exactly the same way as Lemma 4.7. The fact that it is a monoid with identity element $[e]$ follows directly from Lemma A.3 just as in Corollary 6.4. The proof that inverses exist carries over essentially verbatim from the proof of Proposition 6.5 (with slight simplifications, as estimates of the form "$\|a(p^2 - p)\| < \epsilon$" no longer need to be checked). $\square$

Consider now the collection of all pairs (X, ϵ), where X is a finite subset of A_1 and $\epsilon > 0$, made into a directed set as in Definition 6.6. Our goal in the rest of this section is to show that there are isomorphisms

$$KL(A,B) \to \lim_{\leftarrow} KK_\epsilon^{\pi_1,p}(X,B).$$

and

$$\lim_{\leftarrow}{}^1 KK_\epsilon^{\pi_1,p}(X,SB) \to \overline{\{0\}},$$

where $\overline{\{0\}}$ is the closure of $KK(A,B)$. The proof proceeds via the construction of certain intertwining maps.

Definition A.5. Fix notation as in Definition A.2. Provisionally define

$$\phi : \mathcal{P}_\epsilon^{\pi_1,p}(X,B) \to \mathcal{P}_\epsilon^{\pi}(X,B), \quad p \mapsto p + (1 - 1_A)e.$$

Lemma A.6. *The map ϕ from Definition A.7 is well-defined, and descends to a homomorphism*

$$\phi_* : KK_\epsilon^{\pi_1,p}(X,B) \to KK_\epsilon^{\pi}(X,B).$$

Proof. It is straightforward to see that ϕ is a well-defined map that takes homotopies to homotopies and so descends to a well-defined set map $\phi_* : KK_\epsilon^{\pi_1,p}(X,B) \to KK_\epsilon^{\pi}(X,B)$. Let s_1, s_2 be Cuntz isometries inducing the group operation, and define $t_1 := 1_A s_1$ and $t_2 := 1_A s_2$, which are a pair of Cuntz isometries in $\mathcal{L}(1_A E)$ which we may use to define the group operation on $KK_\epsilon^{\pi_1,p}(X,B)$. We compute that for $p, q \in \mathcal{P}_\epsilon^{\pi_1,p}(X,B)$

$$t_1 p t_1^* + t_2 q t_2^* + (1-1_A)e = s_1(p+(1-1_A)e)s_1^* + s_2(q+(1-1_A)e)s_2^*$$

which implies that $\phi_*([p]+[q]) = \phi_*[p] + \phi_*[q]$ as claimed. $\square$

Definition A.7. Fix notation as in Definition A.2. Assume moreover that $\epsilon < 1/8$ and that X contains the unit of A. Let χ be the characteristic function of $(1/2,\infty)$. Provisionally define

$$\psi : \mathcal{P}_\epsilon^{\pi}(X,B) \to \mathcal{P}_{5\sqrt{\epsilon}}^{\pi_1}(X,B), \quad p \mapsto \chi(1_A p 1_A).$$

Lemma A.8. *The map ψ from Definition A.7 is well-defined and descends to a well-defined homomorphism*

$$\psi_* : KK_\epsilon^{\pi}(X,B) \to KK_{5\sqrt{\epsilon}}^{\pi_1,p}(X,B).$$

Proof. First, we check that ψ is well-defined, and takes image where we claim. Let p be an element of $\mathcal{P}_\epsilon^{\pi}(X,B)$. As we are assuming that 1_A is in X, we have that

$$\|[p,1_A]\| < \epsilon. \tag{21}$$

Hence

$$\begin{aligned}\|(1_A p 1_A)^2 - (1_A p 1_A)\| &\leqslant \|1_A p 1_A p - 1_A p\|\|1_A\| \\ &\leqslant \|1_A\|\|[1_A,p]\|\|p\| + \|1_A(p^2-p)\| \\ &< 2\epsilon.\end{aligned}$$

The polynomial spectral mapping theorem thus implies that the spectrum of $1_A p 1_A$ is contained in the $\sqrt{2\epsilon}$-neighbourhood of $\{0,1\}$ in $\mathbb{R}$. As $\epsilon < 1/8$, we have that $\sqrt{2\epsilon} < 1/2$ and so the characteristic function χ of $(1/2,\infty)$ is continuous on the spectrum of $1_A p 1_A$. Hence $\chi(1_A p 1_A)$ makes sense by the continuous functional calculus and moreover

$$\|1_A p 1_A - \chi(1_A p 1_A)\| < \sqrt{2\epsilon}. \tag{22}$$

Hence we see that for any $a \in X$,

$$\|[\chi(1_A p 1_A), a]\| \leqslant \|[\chi(1_A p 1_A) - 1_A p 1_A, a]\| + \|[1_A p 1_A, a]\| < 2\sqrt{2\epsilon} + \epsilon. \tag{23}$$

Putting the discussion so far together, $\chi(1_A p 1_A)$ is a projection in $\mathcal{L}(E)$ such that $\|[\chi(1_A p 1_A), a]\| < 5\sqrt{\epsilon}$ for all $a \in X$. We have moreover that $1_A p 1_A - 1_A e = 1_A(p-e)1_A \in \mathcal{K}(1_A E)$, whence also $\chi(1_A p 1_A) - 1_A e \in \mathcal{K}(1_A E)$. In conclusion, we see that $\chi(1_A p 1_A)$ defines an element of $\mathcal{P}^{\pi_1,p}_{5\sqrt{\epsilon}}(X, B)$. We have thus shown that ψ is well-defined.

It is straightforward to check that homotopies pass through the above construction, so that ψ induces a well-defined map of sets

$$\psi_* : KK^{\pi}_{\epsilon}(X, B) \to KK^{\pi_1,p}_{5\sqrt{\epsilon}}(X, B).$$

Finally, to see that ψ_* is a homomorphism, we fix Cuntz isometries s_1, s_2 inducing the group operation in $KK^{\pi}_{\epsilon}(X, B)$. As in the proof of Lemma A.6, we may use the Cuntz isometries $t_1 := 1_A s_1$ and $t_2 := 1_A s_2$ to define the group operation on $KK^{\pi_1,p}_{5\sqrt{\epsilon}}(X, B)$. Using naturality of the functional calculus and the fact that s_1 and s_2 commute with 1_A, we see that for $p, q \in KK^{\pi}_{\epsilon}(X, B)$ we have that

$$\chi(1_A(s_1 p s_1^* + s_2 q s_2^*)1_A) = t_1 \chi(1_A p 1_A) t_1^* + t_2 \chi(1_A q 1_A) t_2^*,$$

and thus that $\psi_*([p] + [q]) = \psi_*[p] + \psi_*[q]$, completing the proof. $\square$

Lemma A.9. *Fix notation as in Definition A.2. Assume moreover that $\epsilon < 1/8$ and that X contains the unit of A. Consider the diagrams*

$$\begin{tikzcd} KK^{\pi}_{\epsilon}(X, B) \arrow[r, equal] & KK^{\pi}_{\epsilon}(X, B) \arrow[d, "\psi_*"] \\ KK^{\pi_1,p}_{\epsilon}(X, B) \arrow[u, "\phi_*"] \arrow[r] & KK^{\pi_1,p}_{5\sqrt{\epsilon}}(X, B) \end{tikzcd} \tag{24}$$

and

$$\begin{tikzcd}
KK^{\pi}_{\epsilon}(X,B) \arrow[rr] \arrow[dd, "\psi_*"] & & KK^{\pi}_{8\sqrt{\epsilon}}(X,B) \\
& KK^{\pi_1,p}_{5\sqrt{\epsilon}}(X,B) \arrow[ur] & \\
KK^{\pi_1,p}_{5\sqrt{\epsilon}}(X,B) \arrow[r, equal] & KK^{\pi_1,p}_{5\sqrt{\epsilon}}(X,B) \arrow[u, "\phi_*"] &
\end{tikzcd} \tag{25}$$

where the horizontal and diagonal maps are the canonical forget control maps, defined analogously to Remark 6.7, part (v). These both commute.

Proof. For any $p \in \mathcal{P}^{\pi_1,p}_{\epsilon}(X,B)$ we have that $\psi(\phi(p)) = p$, and so the diagram in line (24) clearly commutes. For the diagram in line (25), we need to show that if $p \in \mathcal{P}_{\epsilon}(X,B)$, then the classes of p and of $\chi(1_A p 1_A) + (1-1_A)e$ in $KK_{8\sqrt{\epsilon}}(X,B)$ are the same. For this, we concatenate two homotopies. First, consider the homotopy

$$t \mapsto p_t := \chi(1_A p 1_A) + (1-1_A)(te + (1-t)p), \quad t \in [0,1].$$

As $ap_t = a\chi(1_A p 1_A)$ for all $a \in A$ and all $t \in [0,1]$, we see that $a(p_t^2 - p_t) = 0$. Moreover, as A commutes with e, as $\|[p,a]\| < \epsilon$ for all $a \in X$, and as $\|[\chi(1_A p 1_A), a]\| < 5\sqrt{\epsilon}$ for all $a \in X$, we see that $\|[p_t, a]\| < 5\sqrt{\epsilon} + \epsilon < 6\sqrt{\epsilon}$ for all $a \in X$. Hence this homotopy passes through $\mathcal{P}^{\pi}_{6\sqrt{\epsilon}}(X,B)$, and connects $\chi(1_A p 1_A) + (1-1_A)e$ and $\chi(1_A p 1_A) + (1-1_A)p$.

For the second homotopy, note first that lines (22) and (21) imply that

$$\|\chi(1_A p 1_A) - 1_A p\| \leqslant \|\chi(1_A p 1_A) - 1_A p 1_A\| + \|1_A[1_A, p]\| < \sqrt{2\epsilon} + \epsilon. \tag{26}$$

Consider now the homotopy

$$t \mapsto q_t := (1-t)\chi(1_A p 1_A) + t1_A p + (1-1_A)p, \quad t \in [0,1] \tag{27}$$

Write $r_t := (1-t)\chi(1_A p 1_A) + t1_A p$, so we have $\|r_t - \chi(1_A p 1_A)\| < \sqrt{2\epsilon} + \epsilon$ for all t by line (26). Hence for any $a \in A$,

$$\begin{aligned}
&\|a(q_t^2 - q_t)\| = \|a(r_t^2 - r_t)\| \\
&\quad \leqslant \|r_t(r_t - \chi(1_A p 1_A))\| + \|(\chi(1_A p 1_A) - r_t)\chi(1_A p 1_A)\| + \|r_t - \chi(1_A p 1_A)\| \\
&\quad < 3(\sqrt{2\epsilon} + \epsilon).
\end{aligned}$$

Moreover, for any $a \in X$, lines (23) and (21) give that

$$\|[q_t, a]\| \leqslant \|[\chi(1_A p 1_A), a]\| + \|[1_A p, a]\| + \|[(1 - 1_A)p, a]\| \leqslant \sqrt{5\epsilon} + 2\epsilon.$$

Putting all this together, the homotopy $t \mapsto q_t$ from line (27) passes through $KK^{\pi}_{8\sqrt{\epsilon}}(X, B)$. As this homotopy connects $\chi(1_A p 1_A) + (1 - 1_A)p$ and p, this completes the proof. ☐

We are now in a position to prove the following result.

Proposition A.10. *Let A and B be separable C^*-algebras with A unital. As in Definition 6.6, make the collection of pairs (X, ϵ) with X a finite subset of A_1 and $\epsilon > 0$ into a directed set. Let $\pi : A \to \mathcal{L}(E)$ be a substantially absorbing representation of A on a Hilbert B-module. Then with notation as above there are isomorphisms*

$$KL(A, B) \to \lim_{\leftarrow} KK^{\pi_1, p}_{\epsilon}(X, B).$$

and

$$\lim_{\leftarrow}{}^1 KK^{\pi_1, p}_{\epsilon}(X, SB) \to \overline{\{0\}},$$

where the limits are taken over the directed set of Definition 6.6 and $\overline{\{0\}}$ is the closure of 0 in $KK(A, B)$. Moreover, there is a short exact sequence

$$0 \to \lim_{\leftarrow}{}^1 KK^{\pi_1, p}_{\epsilon}(X, SB) \to KK(A, B) \to \lim_{\leftarrow} KK^{\pi_1, p}_{\epsilon}(X, B) \to 0.$$

Proof. Thanks to Theorems 6.9 and 7.7 respectively, it will suffice to show that

$$\lim_{\leftarrow} KK^{\pi_1, p}_{\epsilon}(X, B) \cong \lim_{\leftarrow} KK^{\pi}_{\epsilon}(X, B) \tag{28}$$

and

$$\lim_{\leftarrow}{}^1 KK^{\pi_1, p}_{\epsilon}(X, B) \cong \lim_{\leftarrow}{}^1 KK^{\pi}_{\epsilon}(X, B). \tag{29}$$

Using Lemmas A.6, A.8, and A.9 we can construct an increasing sequence (X_n) of finite subsets of A_1 with dense union and that all contain the unit, a sequence (ϵ_n) in $(0, 1/8)$ that tends to zero as $n \to \infty$, and a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & KK^{\pi}_{\epsilon_n}(X_n, B) \arrow[r] \arrow[dr, "\psi_*^{(n)}"'] & KK^{\pi}_{\epsilon_{n-1}}(X_{n-1}, B) \arrow[r] \arrow[dr, "\psi_*^{(n-1)}"'] & \cdots \arrow[r] \arrow[dr, "\psi_*^{(2)}"'] & KK^{\pi}_{\epsilon_1}(X_1, B) \\
\cdots \arrow[r] & KK^{\pi_1,p}_{\epsilon_n}(X_n, B) \arrow[r] \arrow[u, "\phi_*^{(n)}"] & KK^{\pi_1,p}_{\epsilon_{n-1}}(X_{n-1}, B) \arrow[r] \arrow[u, "\phi_*^{(n-1)}"] & \cdots \arrow[r] & KK^{\pi_1,p}_{\epsilon_1}(X_1, B) \arrow[u, "\phi_*^{(1)}"']
\end{tikzcd} \tag{30}$$

where: the horizontal maps are forget control maps; the maps labeled $\phi_*^{(n)}$ are from Lemma A.6; the maps labeled $\psi_*^{(n)}$ are from Lemma A.8; each subdiagram of the form

$$\begin{tikzcd}
KK^{\pi}_{\epsilon_n}(X_n, B) \arrow[dr, "\psi_*^{(n)}"] & \\
KK^{\pi_1,p}_{\epsilon_n}(X_n, B) \arrow[r] \arrow[u, "\phi_*^{(n)}"] & KK^{\pi_1,p}_{\epsilon_{n-1}}(X_{n-1}, B)
\end{tikzcd}$$

and each of the form

$$\begin{tikzcd}
KK^{\pi}_{\epsilon_n}(X_n, B) \arrow[r] \arrow[dr, "\psi_*^{(n)}"'] & KK^{\pi}_{\epsilon_{n-1}}(X_{n-1}, B) \\
 & KK^{\pi_1,p}_{\epsilon_{n-1}}(X_{n-1}, B) \arrow[u, "\phi_*^{(n-1)}"']
\end{tikzcd}$$

commutes. Now, by assumption that (X_n) is increasing and has dense union in A_1, and by assumption that $\epsilon_n \to 0$, the sequence (X_n, ϵ_n) is cofinal in the directed set of Definition 6.6. Hence by Remark 6.7 part (vii) and Remark 7.5, the top row of diagram (30) computes $\lim\limits_{\leftarrow (X,\epsilon)} KK^{\pi}_{\epsilon}(X, B)$, while the bottom row computes $\lim\limits_{\leftarrow (X,\epsilon)} KK^{\pi_1,p}_{\epsilon}(X, B)$. The isomorphism in line (28) follows directly from this. The isomorphis in line (29) also follows directly from this and the commuting diagram of line (30), but with B replaced by SB. $\square$

A.2 Unitally absorbing representations

In this subsection, we show a form of representation-independence for the groups from the previous subsection.

First, we recall a definition, which is essentially [25, Definition 2.2] (compare also condition (2) from [25, Theorem 2.1]).

Definition A.11. Let A and B be separable C^*-algebras with A unital, and let F be a Hilbert B-module. A representation $\pi : A \to \mathcal{L}(F)$ is *unitally absorbing* (for the pair (A, B)) if for any Hilbert B-module E and ucp map $\sigma : A \to \mathcal{L}(E)$, there is a sequence (v_n) of isometries in $\mathcal{L}(E, F)$ such that:

(i) $\sigma(a) - v_n^*\pi(a)v_n \in \mathcal{K}(E)$ for all $a \in A$ and $n \in \mathbb{N}$;

(ii) $\|\sigma(a) - v_n^*\pi(a)v_n\| \to 0$ as $n \to \infty$ for all $a \in A$.

The representation π is *strongly unitally absorbing* if it is an infinite amplification of an absorbing representation.

Remark A.12. Let A and B be separable, with A unital as above. Assume also that at least one of A and B is nuclear. It follows from [12, Theorem 5] that if $\pi : A \to \mathcal{B}(\ell^2)$ is a faithful unital representation such that $\pi^{-1}(\mathcal{K}(\ell^2)) = \{0\}$, then the amplification $1 \otimes \pi : A \to \mathcal{L}(\ell^2 \otimes B)$ is unitally absorbing. We will not use this remark in the paper, but it is important for justifying the picture of controlled KK-theory that we use in the introduction.

We will need a lemma relating unitally absorbing representations to strongly absorbing representations.

Lemma A.13. *Say A and B are separable C^*-algebras with A unital, and let $\pi : A \to \mathcal{L}(F)$ be an absorbing representation. Then the corestriction of π_1 of π to a unital representation $\pi_1 : A \to \mathcal{L}(1_A \cdot F)$ is a unitally absorbing representation.*

Proof. Let $\sigma : A \to \mathcal{L}(E)$ be a ucp map with F a Hilbert B-module. As π is absorbing, there is a sequence (v_n) of isometries in $\mathcal{L}(E, F)$ such that

$$\sigma(a) - v_n^*\pi(a)v_n \in \mathcal{K}(E)$$

for all $a \in A$ and $n \in \mathbb{N}$, and such that

$$\|\sigma(a) - v_n^*\pi(a)v_n\| \to 0$$

as $n \to \infty$ for all $a \in A$. As σ is unital we in particular have that $\|1_E - v_n^*\pi(1_A)v_n\| \to 0$ as $n \to \infty$. Set $w_n := \pi(1_A)v_n \in \mathcal{L}(E, \pi(1_A)F) \subseteq \mathcal{L}(E, F)$. We compute that

$$w_n^*w_n - 1_E = v_n^*\pi(1_A)v_n - 1_E$$

so $w_n^*w_n$ is a compact perturbation of 1_E, and $\|w_n^*w_n - 1_E\| \to 0$ as $n \to \infty$. Passing to a subsequence, we may assume in particular that $w_n^*w_n$ is invertible for all n. Note then that

$$\forall n \ (w_n^*w_n)^{-1/2} - 1_E \in \mathcal{K}(E), \quad \text{and} \quad (w_n^*w_n)^{-1/2} - 1_E \to 0 \text{ as } n \to \infty. \quad (31)$$

Define $u_n := w_n(w_n^*w_n)^{-1/2}$. Then (u_n) is a sequence of isometries in $\mathcal{L}(E, \pi(1_A)F)$ such that

$$\sigma(a) - u_n^*\pi(a)u_n = \sigma(a) - (w_n^*w_n)^{-1/2}v_n\pi(a)v_n(w_n^*w_n)^{-1/2}$$

for all $a \in A$. This computation combined with line (31) shows that (u_n) has the properties needed to show that π_1 is unitally absorbing. □

The following corollary is immediate.

Corollary A.14. *Say A and B are separable C^*-algebras with A unital, and let $\pi : A \to \mathcal{L}(F)$ be a strongly absorbing representation on a Hilbert B-module. Then the corestriction of π_1 of π to a unital representation $\pi : A \to \mathcal{L}(1_A \cdot F)$ is a strongly unitally absorbing representation.* □

The following lemma, which follows ideas of Kasparov [12] (compare also [25, Theorem 2.1]), says that unitally absorbing representations are essentially unique.

Lemma A.15. *Say A and B are separable C^*-algebras with A unital, and let $\pi : A \to \mathcal{L}(F)$ and $\sigma : A \to \mathcal{L}(E)$ be unitally absorbing representations. Then there is a sequence (u_n) of unitaries in $\mathcal{L}(E, F)$ such that*

(i) $\sigma(a) - u_n^*\pi(a)u_n \in \mathcal{K}(E)$ *for all* $a \in A$ *and* $n \in \mathbb{N}$*;*

(ii) $\|\sigma(a) - u_n^*\pi(a)u_n\| \to 0$ *as* $n \to \infty$ *for all* $a \in A$*.*

Proof. Let $(\sigma^\infty, E^\infty)$ be the infinite amplification of (σ, E), and let (v_n) be a sequence of isometries in $\mathcal{L}(E^\infty, F)$ such that $v_n^*\pi(a)v_n - \sigma^\infty(a) \to 0$ for all $a \in A$, and such that $v_n^*\pi(a)v_n - \sigma^\infty(a) \in \mathcal{K}(E^\infty)$ for all $a \in A$ and all n. Using that

$$(\pi(a)v_n - v_n\sigma^\infty(a))^*(\pi(a)v_n - v_n\sigma(a)^\infty)$$

equals

$$v_n^*\pi(a^*a)v_n - \sigma^\infty(a^*a) - (v_n^*\pi(a^*)v_n - \sigma^\infty(a^*))\sigma^\infty(a) - \sigma^\infty(a^*)(v_n^*\pi(a)v_n - \sigma^\infty(a))$$

for all n and all $a \in A$, we see that we also have $\pi(a)v_n - v_n\sigma^\infty(a) \in \mathcal{K}(E^\infty, F)$ for all n and all $a \in A$, and that $\|\pi(a)v_n - v_n\sigma^\infty(a)\| \to 0$ as $n \to \infty$ for all $a \in A$.

Now, for representations $\phi : A \to \mathcal{L}(G)$ and $\psi : A \to \mathcal{L}(H)$ on Hilbert B-modules, let us write $\phi \sim \psi$ if there is a sequence of unitaries (u_n) in $\mathcal{L}(G, H)$ such that $\phi(a) - u_n^*\psi(a)u_n \in \mathcal{K}(G)$ for all $a \in A$ and $n \in \mathbb{N}$ and $\|\phi(a) - u_n^*\psi(a)u_n\| \to 0$ as $n \to \infty$ for all $a \in A$. Let $u_n^F \in \mathcal{L}(F, E \oplus F)$ be the unitary built from v_n as in Lemma 2.8. Then the sequence (u_n^F) in $\mathcal{L}(F, E \oplus F)$ shows that $\pi \sim \sigma \oplus \pi$. As the situation is symmetric in σ and π, we also see that $\sigma \sim \sigma \oplus \pi$. As $\sim$ is transitive, we see that $\pi \sim \sigma$ and are done. □

Remark A.16. Using [25, Theorem 2.4], if A and B are separable with A unital, there always exists a unitally absorbing representation $\pi : A \to \mathcal{L}(\ell^2 \otimes B)$. Hence if $\sigma : A \to \mathcal{L}(E)$ is any unitally absorbing representation, we must have that E is isomorphic as a Hilbert B-module to $\ell^2 \otimes B$, i.e. the standard Hilbert B-module $\ell^2 \otimes B$ is the only Hilbert B-module that can admit a unitally absorbing representation. A similar remark applies in the absorbing case, with essentially the same justification.

We will need a unital variant of Definition 4.1.

Definition A.17. A representation $\pi : A \to \mathcal{L}(E)$ is *unitally substantial* if it comes with a fixed grading $(\pi, E) = (\pi_0 \oplus \pi_0, E_0 \oplus E_0)$ such that (π_0, E_0) is strongly unitally absorbing.

Our main goal in this section is the following result, which says essentially that any unitally subs absorbing representation can be used to compute $KL(A, B)$ as an inverse limit. For the statement, let us say that a unitally absorbing representation (π, E) is *balanced graded* if comes with a fixed grading of the form $(\pi_0 \oplus \pi_0, E_0 \oplus E_0)$, with (π_0, E_0) unitally absorbing.

Proposition A.18. *Let A and B be separable C^*-algebras with A unital. Let the collection of pairs (X, ϵ) consisting of a finite subsets X of A_1 and $\epsilon > 0$ be made into a directed set as in Definition 6.6. Then for any unitally substantial representation π of A we have that*

$$KL(A, B) \to \lim_{\leftarrow} KK_{\epsilon}^{\pi,p}(X, B).$$

and

$$\lim_{\leftarrow}{}^1 KK_{\epsilon}^{\pi,p}(X, SB) \to \overline{\{0\}},$$

where $\overline{\{0\}}$ is the closure of 0 in $KK(A, B)$. Moreover, there is a short exact sequence

$$0 \to \lim_{\leftarrow}{}^1 KK_{\epsilon}^{\pi,p}(X, SB) \to KK(A, B) \to \lim_{\leftarrow} KK_{\epsilon}^{\pi,p}(X, B) \to 0.$$

Proof. Let $\kappa : A \to \mathcal{L}(E)$ be a substantial representation as in Definition 4.1 (whence non-unital), with decomposition $\kappa = \kappa_0 \oplus \kappa_0$. Proposition A.10 says (with notation given there) that

$$\lim_{\leftarrow} KK_{\epsilon}^{\kappa_1,p}(X, B) \cong KL(A, B).$$

On the other hand, κ_1 decomposes (with obvious notation) as $\kappa_{0,1} \oplus \kappa_{0,1}$, and $\kappa_{0,1}$ is strongly unitally absorbing by Corollary A.14. Hence κ_1 is unitally substantial. It thus suffices to prove that

$$\lim_{\leftarrow} KK_{\epsilon}^{\pi,p}(X, B) \cong \lim_{\leftarrow} KK_{\epsilon}^{\sigma,p}(X, B)$$

and

$$\lim_{\leftarrow}{}^1 KK_{\epsilon}^{\pi,p}(X, SB) \cong \lim_{\leftarrow}{}^1 KK_{\epsilon}^{\sigma,p}(X, SB)$$

for any unitally substantial representations π and σ.

Let then (π, E) and (σ, F) be unitally substantial representations. For each (X, ϵ), Lemma A.15 gives us a unitary $u = u(X, \epsilon) \in \mathcal{L}(E, F)$ such that $u\pi(a)u^* - \sigma(a) \in \mathcal{K}(F)$ for all $a \in A$, such that $\|u\pi(a)u^* - \sigma(a)\| \leqslant \epsilon$ for all $a \in X$. We may assume also that if e_σ and e_π are the respective neutral elements, then $ue_\pi u^* = e_\sigma$ and that conjugation by u preserves the canonically included copies $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ and $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(F)$ of Lemma 4.2: indeed, this follows by writing each of (π, E) and (σ, F) as a sum of unitally absorbing representations and applying Lemma A.15 to get some unitary u_0, and then amplifying u_0 to get u.

It follows from this and direct checks that conjugation by u gives a well-defined map

$$\mathrm{ad}_u : KK_\epsilon^{\pi,p}(X, B) \to KK_{2\epsilon}^{\sigma,p}(X, B).$$

The situation is symmetric, so we also get

$$\mathrm{ad}_{u^*} : KK_\epsilon^{\pi,p}(X, B) \to KK_{2\epsilon}^{\sigma,p}(X, B).$$

Now, to deduce the existence of an isomorphism $\varprojlim KK_\epsilon^{\pi,p}(X, B) \cong \varprojlim KK_\epsilon^{\sigma,p}(X, B)$ it will suffice to show that the diagram

$$\begin{tikzcd}
KK_\epsilon^{\pi,p}(X, B) \arrow[r] \arrow[d, "\mathrm{ad}_{u(X,\epsilon)}"] & KK_{4\epsilon}^{\pi,p}(X, B) \\
KK_{2\epsilon}^{\sigma,p}(X, B) \arrow[r, equal] & KK_{2\epsilon}^{\sigma,p}(X, B) \arrow[u, "\mathrm{ad}_{u(X,2\epsilon)^*}"']
\end{tikzcd}$$

commutes, where the unlabeled arrow is the canonical forget control map (we also need commutativity of the corresponding diagrams with the roles of σ and π reversed, but this follows by symmetry). The isomorphism $\varprojlim^1 KK_\epsilon^{\pi,p}(X, SB) \cong \varprojlim^1 KK_\epsilon^{\sigma,p}(X, SB)$ will follow similarly on replacing B with SB.

So, we need to show that if $v := u(X, 2\epsilon)^* u(X, 2\epsilon) \in \mathcal{L}(E)$, then ad_v induces the same map $KK_\epsilon^{\pi,p}(X, B) \to KK_{4\epsilon}^{\pi,p}(X, B)$ as the forget control map. Let $s_1, s_2 \in \mathcal{L}(E)$ be Cuntz isometries used to define the group operation. As the neutral element $e = e_\pi$ defines the identity in $KK_\delta^{\pi,p}(X, B)$ for any δ, it suffices to prove that $s_1 v p v^* s_1^* + s_2 e s_2^*$ defines the same element of

$KK^{\pi,p}_{4\epsilon}(X,B)$ as $s_1ps_1^* + s_2es_2^*$. Define $w := s_1vs_1^* + s_2s_2^*$, which is unitary. For $t \in [0,\pi/2]$ define

$$w_t := \cos(t)s_1s_1^* + \sin(t)s_1s_2^* - \sin(t)s_2s_1^* + \cos(t)s_2s_2^*.$$

Note that w_t is unitary and commutes with both A and e. Direct checks (that we leave to the reader) show that

$$w_tww_t^*(s_1ps_1^* + s_2s_2^*)w_tww_t^*, \quad t \in [0,\pi/2]$$

defines a continuous path in $\mathcal{P}^{\pi,p}_{4\epsilon}(X,B)$ that connects $s_1ps_1^* + s_2s_2^*$ and $s_1vpv^*s_1 + s_2s_2^*$, completing the proof. $\square$

Remark A.19. (We thank Claude Schochet for this remark). Let π be a unitally substantial representation of A. Standard separability arguments show that for each $\epsilon > 0$ and finite $X \subseteq A_1$, the group $KK^{\pi,p}_{\epsilon}(X,SB)$ is countable. It follows from an argument of Gray [10, page 242] that a $\lim^1$ group associated to a sequence of countable groups is either zero or uncountable. Hence $\underleftarrow{\lim}^1 KK^{\pi,p}_{\epsilon}(X,SB)$ is either zero or uncountable. Thus Proposition A.18 implies that the closure of 0 in $KK(A,B)$ is always (for A and B separable, and A unital) either zero or uncountable.

A.3 Matricial representations of controlled KK-groups

In this subsection we give a formulation of controlled KK-theory in terms of matrices, which is perhaps closer to standard formulations of elementary C^*-algebra K-theory. Although our main definitions are more convenient for establishing the theory (particularly with regard to the topology on KK), this definition will make computations easier in our subsequent applications.

For a representation $\pi : A \to \mathcal{L}(E)$ we use the amplifications $1_{M_n} \otimes \pi : A \to M_n(\mathcal{L}(E))$ to identify A with a (diagonal) C^*-subalgebra of $M_n(\mathcal{L}(E))$ for all n.

Definition A.20. Let A be unital, and let $\pi : A \to \mathcal{L}(E)$ be a unital representation. Let $\mathcal{K}(E)^+$ be the unitization of $\mathcal{K}(E)$.

Let X be a finite subset of A_1, let $\epsilon > 0$, and let $n \in \mathbb{N}$. Define $\mathcal{P}^{\pi,m}_{n,\epsilon}(X, B)$ to be the collection of pairs (p, q) of projections in $M_n(\mathcal{K}(E)^+)$ satisfying the following conditions:

(i) $\|[p, a]\| < \epsilon$ and $\|[q, a]\| < \epsilon$ for all $a \in X$;

(ii) the classes $[p], [q] \in K_0(\mathbb{C})$ formed by taking the images of p and q under the canonical quotient map $M_n(\mathcal{K}(E)^+) \to M_n(\mathbb{C})$ are the same.

If (p_1, q_1) is an element of $\mathcal{P}^{\pi,m}_{n_1,\epsilon}(X, B)$ and (p_2, q_2) is an element of $\mathcal{P}^{\pi,m}_{n_2,\epsilon}(X, B)$, define

$$(p_1 \oplus p_2, q_1 \oplus q_2) := \left(\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} , \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \right) \in \mathcal{P}^{\pi,m}_{n_1+n_2,\epsilon}(X, B).$$

Define

$$\mathcal{P}^{\pi,m}_{\infty,\epsilon}(X, B) := \bigsqcup_{n=1}^{\infty} \mathcal{P}^{\pi,m}_{n,\epsilon}(X, B),$$

i.e. $\mathcal{P}^{\pi,m}_{\infty,\epsilon}(X, B)$ is the *disjoint* union of all the $\bigsqcup_{n=1}^{\infty} \mathcal{P}^{\pi,m}_{n,\epsilon}(X, B)$.

Equip each $\mathcal{P}^{\pi,m}_{n,\epsilon}(X, B)$ with the norm topology it inherits from $M_n(\mathcal{L}(E)) \oplus M_n(\mathcal{L}(E))$, and equip $\mathcal{P}^{\pi,m}_{\infty,\epsilon}(X, B)$ with the disjoint union topology. Let $\sim$ be the equivalence relation on $\mathcal{P}^{\pi,m}_{\infty,\epsilon}(X, B)$ generated by the following relations:

(i) $(p, q) \sim (p \oplus r, q \oplus r)$ for any element $(r, r) \in \mathcal{P}^{\pi,m}_{\infty,\epsilon}(X, B)$ with both components the same;

(ii) $(p_1, q_1) \sim (p_2, q_2)$ whenever these elements are in the same path component of $\mathcal{P}^{\pi,m}_{\infty,\epsilon}(X, B)$.[22]

Finally, define $KK^{\pi,m}_{\epsilon}(X, B)$ to be $\mathcal{P}^{\pi,m}_{\infty,\epsilon}(X, B)/\sim$.

Lemma A.21. *Let A and B be separable C^*-algebras with A unital. Let $X \subseteq A_1$ be a finite set, and let $\epsilon > 0$. If $\pi : A \to \mathcal{L}(E)$ is any unital representation, then $KK^{\pi,m}_{\epsilon}(X, B)$ is an abelian group.*

[22] Equivalently, both are in the same $\mathcal{P}^{\pi,m}_{n,\epsilon}(X, B)$, and are in the same path component of this set.

Proof. It is clear from the definition that $KK_\epsilon^{\pi,m}(X,B)$ is a monoid with identity element the class $[0,0]$. A standard rotation homotopy shows that $KK_\epsilon^{\pi,m}(X,B)$ is commutative. To complete the proof, we claim that $[q,p]$ is the inverse of $[p,q]$. Indeed, applying a rotation homotopy to the second variable shows that $(p\oplus q, q\oplus p) \sim (p\oplus q, p\oplus q)$, and the element $(p\oplus q, p\oplus q)$ is trivial by definition of the equivalence relation. $\square$

Now, let $\pi : A \to \mathcal{L}(E)$ be a unitally substantial representation as in Definition A.17, i.e. there is a decomposition $(\pi, E) = (\pi_0 \oplus \pi_0, E_0 \oplus E_0)$, where (π_0, E_0) is a strongly unitally absorbing (ungraded) representation. Under this identification, we have a canonical identification $\mathcal{L}(E) = M_2(\mathcal{L}(E_0))$ under which the neutral projection e_π on E corresponds to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Our goal in this section is to establish isomorphisms

$$KL(A,B) \to \lim_{\leftarrow} KK_\epsilon^{\pi_0,m}(X,B)$$

and

$$\lim_{\leftarrow}{}^1 KK_\epsilon^{\pi_0,m}(X,SB) \to \overline{\{0\}}$$

where the limits are (as usual) taken over the directed set of Definition 6.6.

First, we provisionally define

$$\phi : \mathcal{P}_\epsilon^{\pi,p}(X,B) \to \mathcal{P}_{2,\epsilon}^{\pi_0,m}(X), \quad p \mapsto \left(p \ , \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right),$$

where we have used the identification $\mathcal{L}(E) = M_2(\mathcal{L}(E))$ to make sense of the right hand side.

Lemma A.22. *The map ϕ above is well-defined, and descends to a group homomorphism*

$$\phi_* : KK_\epsilon^{\pi,p}(X,B) \to KK_\epsilon^{\pi_0,m}(X,B).$$

Proof. Using the correspondence $e_\pi \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ it is not difficult to see that the image of ϕ is indeed in $\mathcal{P}_{2,\epsilon}^{\pi_0,m}(X,B)$. It is also clear that ϕ takes homotopies to homotopies, so descends to a well-defined map of sets ϕ_* :

$KK_\epsilon^{\pi,p}(X,B) \to KK_\epsilon^{\pi_0,m}(X,B)$. It remains to show that this set map is a homomorphism. For this, let $s_1, s_2 \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ be a pair of Cuntz isometries inducing the operation on $KK_\epsilon^{\pi,p}(X,B)$. For simplicity of notation, let us write $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathcal{K}(E_0)^+)$. Then for $[p],[q] \in KK_\epsilon^{\pi,p}(X,B)$, we see that

$$\phi_*[p+q] = [s_1 p s_1^* + s_2 q s_2^*, e]$$

(the entires on the right should be considered as matrices in $M_2(\mathcal{K}(E_0)^+)$). According to the definition of the equivalence relation defining $KK_\epsilon^{\pi,p}(X,B)$, this is the same element as

$$[s_1 p s_1^* + s_2 q s_2^* \oplus e, e \oplus e].$$

For $t \in [0,\pi/2]$, write

$$u_t := s_1 s_1^* \otimes 1_2 + (s_2 s_2^* \otimes 1_2)(1_2 \otimes \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix})$$

(here $1_2 \in M_2(\mathbb{C})$, so we are considering the element above as an element of $\mathcal{L}(E) \otimes M_2(\mathbb{C}) = M_2(\mathcal{L}(E_0)) \otimes M_2(\mathbb{C})$). Consider now the path

$$(u_t(s_1 p s_1^* + s_2 q s_2^* \oplus e)u_t^*, u_t(e \oplus e)u_t^*), \quad t \in [0,\pi/2]. \tag{32}$$

We have that $u_t(e\oplus e)u_t^* = e \oplus e$ for all t. As $s_1 p s_1^* + s_2 q s_2^* \oplus e - e \oplus e \in M_4(\mathcal{K}(E_0))$, we thus see that

$$M_4(\mathcal{K}(E_0)) \ni u_t(s_1 p s_1^* + s_2 q s_2^* \oplus e)u_t^* - u_t(e\oplus e)u_t^* = u_t(s_1 p s_1^* + s_2 q s_2^* \oplus e)u_t^* - e \oplus e.$$

It follows from this that $u_t(s_1 p s_1^* + s_2 q s_2^* \oplus e)u_t^*$ is in $M_4(\mathcal{K}(E_0)^+)$ for all $t \in [0,\pi/2]$, and therefore the path in line (32) passes through $\mathcal{P}_{4,\epsilon}^{\pi_0,m}(X,B)$. As such, it shows that in $KK_\epsilon^{\pi_0,m}(X,B)$ we have the identity

$$[s_1 p s_1^* + s_2 q s_2^* \oplus e, e \oplus e] = [s_1 p s_1^* + s_2 e s_2^* \oplus s_2 q s_2^* + s_1 e s_1^*, e \oplus e].$$

As the left hand side above is $\phi_*[p+q]$ we thus get

$$\phi_*[p+q] = [s_1 p s_1^* + s_2 e s_2^*, e] + [s_2 q s_2^* + s_2 e s_2^*, e].$$

To complete the proof, it this suffices to show that $[s_1ps_1^* + s_2es_2^*, e] = \phi_*[p]$ and $[s_2qs_2^* + s_2es_2^*, e] = \phi_*[q]$, i.e. that $[s_1ps_1^* + s_2es_2^*, e] = [p, e]$ and $[s_2qs_2^* + s_2es_2^*, e] = [q, e]$. These identities follow directly from Lemma A.3 (the first with $v = s_1$, and the second with $v = s_2$), which completes the proof. $\square$

We now define a map going in the other direction to ϕ, which is unfortunately more complicated. To start, for each n, fix a unitary isomorphism $v_n \in \mathcal{B}(\mathbb{C}^2 \otimes \ell^2, (\ell^2)^{\oplus 2n})$ such that if $p_n : (\ell^2)^{\oplus 2n} \to (\ell^2)^{\oplus 2n}$ is the projection onto the first n components, then $v_n p_n v_n^* = e$, where e is (as usual) the projection of $\mathbb{C}^2 \otimes \ell^2$ onto ℓ^2 arising by projecting $\mathbb{C}^2$ onto its first coordinate. Use the usual (compatible) identifications of E with $\mathbb{C}^2 \otimes \ell^2 \otimes F$ and E_0 with $\ell^2 \otimes F$ for some Hilbert module F, identify $\mathcal{B}(\mathbb{C}^2 \otimes \ell^2, (\ell^2)^{\oplus 2n})$ with a subspace of $\mathcal{L}(E, E_0^{\oplus 2n})$ and consider v_n as an element here. Up to the canonical identification $\mathcal{L}(E_0^{\oplus 2n}) = M_{2n}(\mathcal{L}(E_0))$, we thus see that $v_n M_{2n}(\mathcal{L}(E_0)) v_n^* = \mathcal{L}(E)$, and that

$$v_n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v_n^* = e,$$

where the entries of the matrix on the left are understood as $n \times n$ blocks.

Now, let (p, q) be an element of $\mathcal{P}_{n,\epsilon}^{\pi,m}(X, B)$ for some n. As the images of p and q under the canonical quotient map $\sigma : M_n(\mathcal{K}^+) \to M_n(\mathbb{C})$ are the same in $K_0(M_n(\mathbb{C}))$, there is a unitary $u \in M_n(\mathbb{C})$ such that $\sigma(p) = u\sigma(q)u^*$. Define

$$v := \begin{pmatrix} uqu^* & 1 - uqu^* \\ 1 - uqu^* & uqu^* \end{pmatrix} v_n \in \mathcal{L}(E, E_0^{\oplus 2n}).$$

Provisionally define a map

$$\psi : \mathcal{P}_{\infty,\epsilon}^{\pi_0,m}(X, B) \to \mathcal{P}_{5\epsilon}^{\pi,p}(X, B), \quad (p, q) \mapsto v^* \begin{pmatrix} p & 0 \\ 0 & 1 - uqu^* \end{pmatrix} v.$$

Lemma A.23. *The map ψ above is well-defined, and descends to a group homomorphism*

$$\psi_* : KK_\epsilon^{\pi_0,m}(X, B) \to KK_{5\epsilon}^{\pi,p}(X, B)$$

that does not depend on the choice of u or v_n.

Proof. We first have to see that ψ takes image where we say. For simplicity of notation, let us replace q with uqu^*, so we have that $p-q$ is in $M_n(\mathcal{K})$. Hence

$$\begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1-q \end{pmatrix}\begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} \\ - \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & 1-q \end{pmatrix}\begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}$$

is in $M_{2n}(\mathcal{K})$, or in other words

$$\begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1-q \end{pmatrix}\begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is in $M_{2n}(\mathcal{K})$. Conjugating by v_n, and identifying $\mathcal{L}(E) = M_2(\mathcal{L}(E_0))$ with the top left corner of $M_{2n}(\mathcal{L})$, and also recalling the correspondence $e_\pi \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we see that

$$v_n^* \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1-q \end{pmatrix}\begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} v_n - e_\pi$$

is in $M_2(\mathcal{K}(E_0))$. It is moreover not difficult to see that the projection

$$\begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1-q \end{pmatrix}\begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}$$

commutes with elements of X up to error 5ϵ. At this point, we have that ψ takes image where we claimed, so indeed does define a function $\psi : \mathcal{P}^{\pi_0,m}_{\infty,\epsilon}(X,B) \to \mathcal{P}^{\pi,p}_{5\epsilon}(X,B)$.

We now pass to the quotient on the right hand side, so getting a map $\psi_\flat : \mathcal{P}^{\pi_0,m}_{\infty,\epsilon}(X,B) \to KK^{\pi,p}_{5\epsilon}(X,B)$. We will show that this map does not depend on the choice of u or v_n, which will certainly imply the same thing for ψ_* once we show the latter exists. To see that $\psi_\flat$ does not depend on the choices of u such that $\sigma(p) = u\sigma(q)u^*$, note that if $U_n(\mathbb{C})$ is the unitary group of $M_n(\mathbb{C})$, then the collection of all such unitaries is homeomorphic to $\sigma(p)U_n(\mathbb{C})\sigma(q) \times (1-\sigma(p))U_n(\mathbb{C})(1-\sigma(q))$, so path connected. Hence any two such choices give rise to homotopic elements of $\mathcal{P}^{\pi,p}_{5\epsilon}(X,B)$. One can argue that $\psi_\flat$ does not depend on the choice of v_n similarly: any two such choices are connected by a path that passes through such elements.

We now show that ψ descends to a well-defined map $\psi_* : KK^{\pi_0,m}_\epsilon(X,B) \to KK^{\pi,p}_{5\epsilon}(X,B)$. First we look at part (ii) of the definition of the equivalence relation defining $KK^{\pi_0,m}_\epsilon(X,B)$, so let $(p_t,q_t)_{t\in[0,1]}$ be a homotopy in some $\mathcal{P}^{\pi_0,m}_{n,\epsilon}(X,B)$. Using for example Lemma 4.8 we may choose a continuous path of unitaries $(u_t)_{t\in[0,1]}$ in $M_n(\mathbb{C})$ such that $\sigma(p_t) = u_t\sigma(q)u_t^*$ for all $t \in [0,1]$, and use these to define $\phi_*[p_t,q_t]$ for each t. From here, it is straightforward to see that ψ takes homotopies to homotopies, so we are done with this part of the equivalence relation.

For part (i) of this equivalence relation, we compute the image of $(p \oplus r, q \oplus r)$ under ψ as follows, where $p,q \in M_n(\mathcal{K}(E_0)^+)$ and $r \in M_k(\mathcal{K}(E_0)^+)$ for some $n,k \in \mathbb{N}$. Let $u \in M_n(\mathbb{C})$ be a unitary such that $\sigma(p) = u\sigma(q)u^*$ in $M_n(\mathbb{C})$, and set $q' = uqu^*$. Then one computes that ψ sends $(p\oplus r, q\oplus r)$ to

$$v_{n+k}^* \begin{pmatrix} q'pq'+1-q' & 0 & q'p(1-q') & 0 \\ 0 & 1 & 0 & 0 \\ (1-q')pq' & 0 & (1-q')p(1-q') & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} v_{n+k} \tag{33}$$

(the odd rows (respectively, columns) have height (resp. width) n, and the even rows (resp. columns) have height (resp. width) k). On the other hand, ψ sends (p,q) to

$$v_n^* \begin{pmatrix} q'pq'+1-q' & q'p(1-q') \\ (1-q')pq' & (1-q')p(1-q') \end{pmatrix} v_n, \tag{34}$$

so we must show that the elements in lines (33) and (34) define the same class in $KK^{\pi,p}_{5\epsilon}(X,B)$. Let now $i : E_0^{\oplus 2n} \to E_0^{\oplus 2(n+k)}$ be the canonical inclusion, and let $w_n := i\circ v_n \in \mathcal{B}(\mathbb{C}^2\otimes\ell^2, (\ell^2)^{\otimes 2(n+k)}) \subseteq \mathcal{L}(E, E_0^{2(n+k)})$. Set $v := v_{n+k}^* w_n$, which is an isometry in $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$. Looking back at line (33), we have that

$$v_{n+k}^* \begin{pmatrix} q'pq'+1-q' & 0 & q'p(1-q') & 0 \\ 0 & 1 & 0 & 0 \\ (1-q')pq' & 0 & (1-q')p(1-q') & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} v_{n+k}$$

$$= v_{n+k}^* \begin{pmatrix} q'pq'+1-q' & 0 & q'p(1-q') & 0 \\ 0 & 0 & 0 & 0 \\ (1-q')pq' & 0 & (1-q')p(1-q') & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} v_{n+k} + v_{n+k}^* \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} v_{n+k}.$$

The terms on the left and right above are equal to

$$vv_n^* \begin{pmatrix} q'pq'+1-q' & q'p(1-q') \\ (1-q')pq' & (1-q')p(1-q') \end{pmatrix} v_n v^* \quad \text{and} \quad (1-vv^*)e$$

respectively. Putting all this together, we see that

$$\psi_*[p \oplus r, q \oplus r] = \left[vv_n^* \begin{pmatrix} q'pq'+1-q' & q'p(1-q') \\ (1-q')pq' & (1-q')p(1-q') \end{pmatrix} v_n v^* + (1-vv^*)e \right].$$

Lemma A.3 implies that the class on the right equals the class of the element in line (34), however, so we are done with this case of the equivalence relation too.

At this point, we know that $\psi_* : KK_\epsilon^{\pi_0,m}(X,B) \to KK_{5\epsilon}^{\pi,p}(X,B)$ is a well-defined set map. It remains to show that ψ_* is a group homomorphism. Let then (p_1,q_1) and (p_2,q_2) be elements of $\mathcal{P}_{n_1,\epsilon}^{\pi_0,m}(X,B)$ and $\mathcal{P}_{n_2,\epsilon}^{\pi_0,m}(X,B)$ respectively. For notational simplicity, assume $p_i - q_i \in M_{n_i}(\mathcal{K}(E_0))$ for $i \in \{1,2\}$ by conjugating by an appropriate unitary as in the definition of ψ; this makes no real difference to the computations below. The sum $[p_1,q_1]+[p_2,q_2]$ is represented by $[p_1 \oplus p_2, q_1 \oplus q_2]$, and ψ_*, this is mapped to the class of the product

$$v_{n_1+n_2}^* \begin{pmatrix} q_1 & 0 & 1-q_1 & 0 \\ 0 & q_2 & 0 & 1-q_2 \\ 1-q_1 & 0 & q_1 & 0 \\ 0 & 1-q_2 & 0 & q_2 \end{pmatrix} \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & 1-q_1 & 0 \\ 0 & 0 & 0 & 1-q_2 \end{pmatrix}$$
$$\cdot \begin{pmatrix} q_1 & 0 & 1-q_1 & 0 \\ 0 & q_2 & 0 & 1-q_2 \\ 1-q_1 & 0 & q_1 & 0 \\ 0 & 1-q_2 & 0 & q_2 \end{pmatrix} v_{n_1+n_2}. \tag{35}$$

Let now s be the permutation unitary in $\mathcal{B}((\ell^2)^{\oplus 2(n_1+n_2)}) \subseteq \mathcal{L}(E_0^{\oplus 2(n_1+n_2)})$ such that conjugation by s flips the second and third rows and columns in the matrices above. Let $w_1 := i_{n_1} v_{n_1}$, where $i_{n_1} : E_0^{\oplus 2n_1} \to E_0^{\oplus 2(n_1+n_2)}$ is the natural inclusion, and similarly for w_2. Set $s_1 := v_{n_1+n_2}^* s w_1$ and $s_2 := v_{n_1+n_2}^* s w_2$, so $s_1, s_2 \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$, and satisfy the Cuntz relation $s_1 s_1^* + s_2 s_2^*$ (this follows as $w_1 w_1^* + w_2 w_2^* = 1$). According to Lemma A.4, we may use s_1 and s_2 to define the group operation on $KK_{5\epsilon}^{\pi,p}(X, B)$, and so

$$\begin{aligned} &\psi_*[p_1, q_1] + \psi_*[p_2, q_2] \\ &= \Bigg[s_1 v_{n_1}^* \begin{pmatrix} q_1 & 1-q_1 \\ 1-q_1 & q_1 \end{pmatrix} \begin{pmatrix} p_1 & 0 \\ 0 & 1-q_1 \end{pmatrix} \begin{pmatrix} q_1 & 1-q_1 \\ 1-q_1 & q_1 \end{pmatrix} v_{n_1} s_1^* \\ &\quad + s_1 v_{n_2}^* \begin{pmatrix} q_2 & 1-q_2 \\ 1-q_2 & q_2 \end{pmatrix} \begin{pmatrix} p_2 & 0 \\ 0 & 1-q_2 \end{pmatrix} \begin{pmatrix} q_2 & 1-q_2 \\ 1-q_2 & q_2 \end{pmatrix} v_{n_2} s_2^* \Bigg]. \end{aligned}$$

A direct computation shows that this equals the element in line (35) above, however, so we are done. □

We need one more technical lemma before we get to the main point.

Lemma A.24. *Let A and B be separable C^*-algebras with A unital. Let $\pi : A \to \mathcal{L}(E)$ be a unitally substantial representation of A on a Hilbert*

B-module. Let $X \subseteq A_1$ be finite, and let $\epsilon > 0$. Consider the diagrams

$$\begin{tikzcd} KK_{\epsilon}^{\pi,p}(X,B) \arrow[r] \arrow[d, "\phi_*"] & KK_{5\epsilon}^{\pi,p}(X,B) \\ KK_{\epsilon}^{\pi_0,m}(X,B) \arrow[r, equal] & KK_{\epsilon}^{\pi_0,m}(X,B) \arrow[u, "\psi_*"'] \end{tikzcd} \tag{36}$$

and

$$\begin{tikzcd} KK_{5\epsilon}^{\pi,p}(X,B) \arrow[r, equal] & KK_{5\epsilon}^{\pi_0,p}(X,B) \arrow[d, "\phi_*"] \\ KK_{\epsilon}^{\pi_0,m}(X,B) \arrow[r] \arrow[u, "\psi_*"] & KK_{5\epsilon}^{\pi_0,m}(X,B) \end{tikzcd} \tag{37}$$

where the horizontal arrows are the canonical forget control maps. These commute.

Proof. We first look at diagram (36). We compute that for $p \in \mathcal{P}_{\epsilon}^{\pi,p}(X,B)$,

$$\psi\phi(p) = v_2^* \begin{pmatrix} e & 1-e \\ 1-e & e \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1-e \end{pmatrix} \begin{pmatrix} e & 1-e \\ 1-e & e \end{pmatrix} v_2.$$

The entries appearing above can be identified with 2×2 matrices, with diagonal matrix units corresponding to e and $1-e$, and p corresponding to the matrix $\begin{pmatrix} epe & ep(1-e) \\ (1-e)pe & (1-e)p(1-e) \end{pmatrix}$. With respect to this picture, one computes that the above equals

$$v_2^* \begin{pmatrix} epe & 0 & 0 & ep(1-e) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1-e)pe & 0 & 0 & (1-e)p(1-e) \end{pmatrix} v_2.$$

Define

$$i : E_0^{\oplus 2} \to E_0^{\oplus 4}, \quad (v,w) \mapsto (v,0,0,w),$$

and define $v := v_2^* i$, which is an isometry inside the copy $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ from Lemma 4.2. One computes using the above that

$$\psi\phi(p) = vpv^* + (1-vv^*)e,$$

whence $[\psi\phi(p)] = [p]$ by Lemma A.3, as required.

Now let us look at diagram (37). Let (p, q) be an element of $\mathcal{P}^{\pi_0, m}_{n,\epsilon}(X, B)$ for some n. For notational convenience, assume that $p - q \in M_n(\mathcal{K}(E_0))$; this can be achieved by conjugating by a unitary in $M_n(\mathbb{C})$, and helps streamline notation below, while making no real difference to the argument. Again adopting the notation e for $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we compute that

$$\phi\psi(p, q) = \left(v_n^* \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1-q \end{pmatrix} \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} v_n \, , \, e \right).$$

Let 0_{2n-2} be the zero element in $M_{2n-2}(\mathcal{K}(E_0)^+)$. Then the element above has the same class as

$$\left(v_n^* \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1-q \end{pmatrix} \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} v_n \oplus 0_{2n-2} \, , \, e \oplus 0_{2n-2} \right). \tag{38}$$

Now, let $p_n : E_0^{\oplus 2n} \to E_0^{\oplus 2}$ be defined by projecting onto the first two coordinates, and define $w_n := v_n p_n$, which is a co-isometry in $\mathcal{B}((\ell^2)^{\oplus 2n}) \subseteq \mathcal{L}(E_0^{\oplus n})$ with source projection the projection onto the first two coordinates in $E_0^{\oplus 2n}$. The space of all co-isometries in $\mathcal{B}((\ell^2)^{\oplus 2n})$ with source projection dominating the projection onto the first two coordinates is connected[23] (in the norm topology). Hence we may connect w_n through such co-isometries to one that acts as the identity on the first two coordinates, from which it follows that the element in line (38) represents the same class as

$$\left(w_n \left(v_n^* \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1-q \end{pmatrix} \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} v_n \oplus 0_{2n-2} \right) w_n^* \, , \right.$$
$$\left. w_n (e \oplus 0_{2n-2}) w_n^* \right).$$

[23] If $n = 1$, such co-isometries are automatically unitary, but this is still a norm-connected space.

Computing, this equals

$$\left(\begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1-q \end{pmatrix} \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right), \tag{39}$$

where all blocks in the matrices appearing above are $n \times n$. Note now that the matrix $\begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}$ whence if we write

$$r := \frac{1}{2}\left(\begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

we see that r is a projection and that $\|[r,a]\| < \epsilon$ for all $a \in X$. For $t \in [0,\pi]$ define $u_t := r + \exp(it)(1-r)$, so (u_t) is a path of unitaries connecting $\begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}$ to the identity, and all (u_t) satisfy $\|[u_t,a]\| < \epsilon$ for all $a \in X$. Hence the path

$$\left(u_t \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1-q \end{pmatrix} \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} u_t^*, u_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u_t^* \right)$$

shows that the element in line (39) defines the same class as

$$\left(\begin{pmatrix} p & 0 \\ 0 & 1-q \end{pmatrix}, \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} \right),$$

which equals $(p \oplus 1-q, q \oplus 1-q)$. This last element defines the same class as (p,q) by definition, however, so we are done. □

Proposition A.25. *Let A and B be separable C^*-algebras with A unital. Let $\pi : A \to \mathcal{L}(E)$ be a unitally substantial representation of A on a Hilbert B-module. Then there are isomorphisms*

$$KL(A,B) \to \lim_{\leftarrow} KK_\epsilon^{\pi_0,m}(X,B).$$

and

$$\lim_{\leftarrow}{}^1 KK_\epsilon^{\pi_0,m}(X,SB) \to \overline{\{0\}},$$

where the limits are taken over the directed set of Definition 6.6 and $\overline{\{0\}}$ *is the closure of* 0 *in* $KK(A,B)$. *Moreover, there is a short exact sequence*

$$0 \to \varprojlim{}^1 KK_\epsilon^{\pi_0,m}(X,SB) \to KK(A,B) \to \varprojlim KK_\epsilon^{\pi_0,m}(X,B) \to 0.$$

Proof. The proof follows from Lemma A.24, quite analogously to that of Proposition A.10. We leave the details to the reader. □

Let us conclude with a final lemma on representation-independence, which is an analogue of Proposition A.18 above.

Lemma A.26. *Let* A *and* B *be separable* C^**-algebras with* A *unital. Let the collection of pairs* (X,ϵ) *consisting of a finite subsets* X *of* A_1 *and* $\epsilon > 0$ *be made into a directed set as in Definition 6.6. Then for any unitally absorbing representation* $\pi : A \to \mathcal{L}(E)$ *we have that*

$$KL(A,B) \to \varprojlim KK_\epsilon^{\pi,m}(X,B).$$

and

$$\varprojlim{}^1 KK_\epsilon^{\pi,m}(X,SB) \to \overline{\{0\}},$$

where $\overline{\{0\}}$ *is the closure of* 0 *in* $KK(A,B)$. *Moreover, there is a short exact sequence*

$$0 \to \varprojlim{}^1 KK_\epsilon^{\pi,m}(X,SB) \to KK(A,B) \to \varprojlim KK_\epsilon^{\pi,m}(X,B) \to 0.$$

Proof. Proposition A.25 tells us that the result is true for *some* unitally absorbing representation. Hence just as in the proof of Proposition A.18 above, it suffices to prove that for any two unitally absorbing representations (π,E), (σ,F), we have that

$$\varprojlim KK_\epsilon^{\pi,m}(X,B) \cong \varprojlim KK_\epsilon^{\sigma,m}(X,B)$$

and similarly for the $\lim^1$-groups. This follows by an intertwining argument based on the unitaries from Lemma A.15, just as in the proof of Proposition A.18: we leave the details to the reader. □

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